Atomicity of free algebras and Gödel's first incompleteness Theorem *Part2: The incompleteness Theorem algebraically for L_n and its guarded versions; non atomicity of the free algebras*

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Abstract . This is a survey article on Gödel's incompleteness theorem for finite variable fragments of first order logic and some of their guarded fragments, approached algebraically. We relate atomicity of free cylindric-like algebras to Gödel's incompleteness result. Full proofs are included for certain special cases carefully selected to give the gist of the idea of methods used. Proofs given are entirely complete. We aspire to present our methods in a way that is accessible to the non-specialist and informative to the practitioner reaching boundaries of current research in a topic that we find fascinating and potentially on the mathematical level, with endless philosophical repercussions. As far as our technical proofs are concerned, Let $3 < m < \omega$, $2 < n < \omega$ and β be any finite cardinal. We show that for any class K between CA_m and RCA_m and the free K algebras on β generators $\mathfrak{Fr}_\beta\mathsf{K}$ is not atomic. We show that the freely generated diagonal free reducts of $\mathsf{Crs_n}$, with finitely many generators, are not atomic either. We survey and elaborate on stronger analogous results obtained by Hajnal Andréka , István Németi and Mohamed Khaled on cylindric-like of dimension n possibly non-commutative algebras. This gives a family of guarded fragments of finite variable first order logic, that are decidable, but enjoys a form of Gödel's incompleteness Theorem, an indeed telling co-existence. After all, Gödel's incompleteness Theorems in several guarded logics, and 'undecidability of the logic at hand' do not necessarily go hand in hand.

1 Introduction

1.1 Gödel's Theorems algebraically

For a class K of algebras, and a cardinal $\beta > 0$, $\mathfrak{Fr}_\beta K$ stands for the β-generated free K algebra. In particular, $\mathfrak{Fr}_\beta \mathsf{CA}_n$ denotes the β-generated free cylindric algebra of dimension n. The following is known: If $\beta \geq \omega$, Pigozzi proved that $\mathfrak{Fr}_\beta \mathsf{CA}_n$ is atomless (has no atoms) [12, Theorem 2.5.13]. Assume that $0 < \beta < \omega$. If $n < 2$, then $\mathfrak{Fr}_\beta \mathsf{CA}_n$ is finite, hence atomic, [12, Theorem 2.5.3 (i)]. $\mathfrak{Fr}_\beta \mathsf{CA}_2$ is infinite but still atomic, a result of Henkin's [12, Theorems 2.5.3 (ii), .4.7 (ii)]. If $3 \leq n < \omega$, Tarski proved that $\mathfrak{Fr}_\beta \mathsf{CA}_n$ has infinitely many atoms [12, Theorem 2.5.9], and it was posed as an open question, cf. [12, Problem 2.14] whether it is atomic or not. Recent research in algebraic logic has revealed however that some guarded fragments of first order logic, surviving undecidability and other undesirable properties, made contact with a deep and subtle *algebraic reformulation* of Gödel's incompleteness theorem. This was proved first by Németi for the 'unguarded' L_n ($n \geq 3$) with square Tarskian semantics by translating a form of Gódel's incompleteness property to non-atomicity of the free algebras and then proving for any finite m, $\mathfrak{Fr}_m \mathsf{CA}_n$ is not atomic. The key idea is that if T is a finite consistent complete L_n theory, then the equivalence class of $\bigwedge T$ will be an atom in the formula algebra of pure logic, built up of the symbols occurring in formulas in T. Here completeness and consistency are defined with respect to provability using only n variables. Now if one finds a formula that *cannot* be extended to such a theory then there will be no atoms below the equivalence class of this formula, and here is where Gödel's Theorem intervenes. The idea in Német's proof uses a translation function of $L_{\omega,\omega}$ into L_3 together with the pairing technique of Tarski's suitably re-defined to adapt the case of three variables rather than four; the latter being the natural habitat of relation algebras [37]. Here we extend such results to guarded fragments [15].

Using similar but more sophisticated methods, in [37], Andréka and Németi later proved the analogous result for the $\mathfrak{Fr}_m \mathsf{Df_n}$ free algebras, solving a long standing open problem in algebraic logic, posed by Tarski, Maddux, Németi and others. This was deduced from the fact that the whole of ZFC can be coded in Df_3; in other words, Df_3, which is substantially weaker than CA_3 *a fortiori* strictly weaker than the calculus of relations is an adequate vehicle for the whole of mathematics. Tarski and Givant [35] had formalized set theory in the calculus of relations establishing an intriguing 'variable free' approach to meta-mathematics. Their joint work in this fascinating topic is published in the monograph *A formalization of Set Theory without variables.* In *op.cit* it is shown that in principle mathematics can be developed in the very simple framework of equations and substitution of equals for equals rather than the customary basis using set theory formalized in first order logic, which is, to say the least, an impressive tour de force with profound meta-mathematical and philosophical repercussions. The first chapter of Andréka et al [4] gives an excellent account of these results.

1.2 Gödel's theorem for guarded fragments; an algebraic approach

Fix $n < \omega$. The notion of guarding syntax accompanied by relativizing semantics can be traced back to the classical Andréka–Resek–Thompson result [5] reproved in [31, 18] using games, which says that every n–dimensional algebra that has the same signature as CA_n, satisfying a certain finite set of equations together with the the so–called *merry go round identities*, is representable by set algebras whose top elements are *diagonizable* in the following sense: If V $(\subseteq {}^nU)$ is the top element of a given set algebra of dimension n, then $s \in V \implies s\circ[i|j] \in V$. We say that $V \subseteq {}^nU$ is *locally square* if whenever $s \in V$ and $\tau : n \to n$, then $s \circ \tau \in V$. Let D_n (G_n) be the class of set algebras whose top elements are diagonizable (locally square) and operations are defined like cylindric set algebra of dimension n relativized to the top element V (the guard).

Theorem 1.1. *[5, 18, 3]. Fix $2 < n < \omega$. Then D_n and G_n are finitely axiomatizable and have a decidable universal (hence equational) theory.*

The philosophy of guarding is to try and tame the wild and often unruly behavior of L_n with standard semantics which manifests itself in a long list of negative results for RCA_n to name a few: Infinite axiomatizability, any equational axiomatization must contain infinitely many variables and infinitely many non-canonical equations, *a fortiori* non-Sahlqvist ones, undecidability of telling whether a finite CA_n has a representation ..., and the list goes on [23, 22, 30, 1, 31]. Substantial effort has gone into proving negative results about what kind of axiomatization are possible for RCA_n. The lesson of these many years' work (appearing in dozens of publications) is presumably that any such axiomatization is bound to be complicated provoking conflicting research interdisciplinary between mathematical and philosophical logic to find out algebraizations of first order logic whose algebraic counterpart are enlightening and simple to describe. Using finitely presented semigroups [29, 31] and/or guarding [2] is a fruitful outcome of the last long and winding road of research. The core of the idea of guarding is to find a semantics that give just the right action while additional effects of square set–theoretic representations are separated out as negotiable decisions of formulation that can threaten completeness and decidability. Using square semantics is a voluntary commitment to one particular mathematical implication whose complexity seems to be an overkill. An insidious term that often confuses this issue is the 'concreteness of set theoretic models' and the pre–assumption of the canonicity of 'simple' square ones; guarding intriguingly reveals the exact opposite. The square ones are the most complicated 'concealing' many far better well behaved multi-modal logics. In the guarded fragment, put forward by Andréka, Van Benthem and Németi, one looks at quantification patterns. Only relativized quantification (along the accessibility relation of the Kripke frame) is allowed. In modal formula of the guarded fragment complexity results deciding validity of sentences is complete for double exponential time, but the n-variable fragments of the guarded fragments are EXPTIME complete, and some 2-variable fragments are even in PSPACE, cf. [6] for a thorough overview.

2 Preliminaries: Cylindric algebras

A *cylindric algebra* consists of a Boolean algebra endowed with an additional structure consisting of distinguished elements and operations, satisfying a certain system of equations. The introduction and study of these algebras has its motivation in two parts of mathematics: the deductive systems of first-order logic, and a portion of elementary set theory dealing with spaces of various dimensions, better known as cylindric set algebras; such algebras also have a geometric twist, reflected in the terminology 'cylinder'. If we are working in 3 dimensions, and we apply the unary operations of cylindrifiers (algebraizing existential quantifiers) to a 'circle', then we are forming the cylinder based on this circle.

Cylindric set algebras are algebras whose elements are relations of a certain pre-assigned arity, endowed with set–theoretic operations that utilize the form of elements of the algebra as sets of sequences. For a set V, $\mathcal{B}(V)$ denotes the Boolean set algebra $\langle \wp(V), \cup, \cap, \sim, \emptyset, V\rangle$. Let U be a set and α an ordinal; α will be the dimension of the algebra. For $s, t \in {}^{\alpha}U$ write $s \equiv_i t$ if $s(j) = t(j)$ for all $j \neq i$. For $X \subseteq {}^{\alpha}U$ and $i, j < \alpha$, let

$$\mathsf{C}_i X = \{s \in {}^{\alpha}U : \exists t \in X(t \equiv_i s)\}$$

and

$$\mathsf{D}_{ij} = \{s \in {}^{\alpha}U : s_i = s_j\}.$$

$\langle \mathcal{B}({}^{\alpha}U), \mathsf{C}_i, \mathsf{D}_{ij}\rangle_{i,j<\alpha}$ is called *the full cylindric set algebra of dimension* α with unit (or greatest element) ${}^{\alpha}U$. Examples of subalgebras of such set algebras arise naturally from models of first order theories. Indeed, if M is a first order structure in a first order signature L with α many variables, then one manufactures a cylindric set algebra based on M as follows. Let

$$\phi^{\mathsf{M}} = \{s \in {}^{\alpha}\mathsf{M} : \mathsf{M} \models \phi[s]\},$$

(here $\mathsf{M} \models \phi[s]$ means that s satisfies ϕ in M), then the set $\{\phi^{\mathsf{M}} : \phi \in Fm^L\}$ is a cylindric set algebra of dimension α, where Fm^L denotes the set of first order formulas taken in the signature L. To see why, we have:

$$\begin{aligned} \phi^{\mathsf{M}} \cap \psi^{\mathsf{M}} &= (\phi \wedge \psi)^{\mathsf{M}}, \\ {}^{\alpha}\mathsf{M} \sim \phi^{\mathsf{M}} &= (\neg\phi)^{\mathsf{M}}, \\ \mathsf{C}_i(\phi^{\mathsf{M}}) &= \exists v_i \phi^{\mathsf{M}}, \\ \mathsf{D}_{ij} &=: (x_i = x_j)^{\mathsf{M}}. \end{aligned}$$

Following [12], Cs_α denotes the class of all subalgebras of full set algebras of dimension α. The (equationally defined) CA_α class is obtained from cylindric set algebras by a process of abstraction and is defined by a *finite* schema of equations given in [12, Definition 1.1.1] that holds of course in the more concrete set algebras. (This is soundness condition).

Definition 2.1. Let α be an ordinal. By *a cylindric algebra of dimension* α, briefly a CA_α, we mean an algebra

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, \mathsf{c}_i, \mathsf{d}_{ij} \rangle_{\kappa,\lambda<\alpha}$$

where $\langle A, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra such that $0, 1$, and d_{ij} are distinguished elements of A (for all $j, i < \alpha$), $-$ and c_i are unary operations on A (for all $i < \alpha$), $+$ and . are binary operations on A, and such that the following equations are satisfied for any $x, y \in A$ and any $i, j, \mu < \alpha$:

(C_1) $\mathsf{c}_i 0 = 0$,

(C_2) $x \leq \mathsf{c}_i x$ (*i.e.*, $x + \mathsf{c}_i x = \mathsf{c}_i x$),

(C_3) $\mathsf{c}_i(x \cdot \mathsf{c}_i y) = \mathsf{c}_i x \cdot \mathsf{c}_i y$,

(C_4) $\mathsf{c}_i \mathsf{c}_j x = \mathsf{c}_j \mathsf{c}_i x$,

(C_5) $\mathsf{d}_{ii} = 1$,

(C_6) if $i \neq j, \mu$, then $\mathsf{d}_{j\mu} = \mathsf{c}_i(\mathsf{d}_{ji} \cdot \mathsf{d}_{i\mu})$,

(C_7) if $i \neq j$, then $\mathsf{c}_i(\mathsf{d}_{ij} \cdot x) \cdot \mathsf{c}_i(\mathsf{d}_{ij} \cdot -x) = 0$.

For operators on classes of algebras, $\mathbf{S}$ stands for the operation of forming subalgebras, $\mathbf{P}$ stands for that of forming products, and $\mathbf{H}$ stands for the operation of forming homomorphic images. The variety of representable algebras of dimension α, α an ordinal, for all such classes are defined as follows: $\mathsf{RCA}_\alpha = \mathbf{SP}\mathsf{Cs}_\alpha$, An algebra $\mathfrak{A} \in \mathsf{CA}_\omega$ is *locally finite*, if the dimension set of every element $x \in A$ is finite. The dimension set of x, or Δx for short, is the set $\{i \in \omega : \mathsf{c}_i x \neq x\}$. Locally finite algebras correspond to Tarski–Lindenbaum algebras of (first order) formulas; in such algebras the dimension set of (an equivalence class of) a formula reflects the number of (finite) set of free variables in this formula. Tarski proved that every locally finite ω-dimensional cylindric algebra is representable, i.e. isomorphic to a subdirect product of set algebra each of dimension ω. The representation theorem $\mathsf{Lf}_\omega \subseteq \mathsf{RCA}_\omega$ is non-trivial; in fact it is equivalent to Gödel's celebrated Completeness Theorem [12, §4.3]. Completeness in the general case is a huge subject that has provoked extensive research.

The restrictive character of the dimension ω and local finiteness were removed in the early course of the development of the subject, and the class CA_α, of cylindric algebras of dimension α, where α is any ordinal, finite or transfinite, was introduced. Three pillars in the development of the subject, and even one can say *the* three pillars in the development of the subject are Tarski's representability result of locally finite algebras, Henkin's characterization of the variety of representable algebras of any dimension via neat embeddings, in his celebrated Neat Embedding Theorem [12, Theorem 3.2.10], and Monk's proof that the variety of representable algebras of dimension > 2 cannot be axiomatized by a finite schema [22]. The last two results involve the central notion of neat reducts:

Definition 2.2. Let $\alpha < \beta$ be ordinals and $\mathfrak{B} \in \mathsf{CA}_\beta$. Then the *$\alpha$–neat reduct* of $\mathfrak{B}$, in symbols $\mathsf{Nr}_\alpha\mathfrak{B}$, is the algebra obtained from $\mathfrak{B}$, by discarding cylindrifiers and diagonal elements whose indices are in $\beta \sim \alpha$, and restricting the universe to the set $Nr_\alpha B = \{x \in \mathfrak{B} : \{i \in \beta : \mathsf{c}_i x \neq x\} \subseteq \alpha\}$.

Let α be any ordinal. If $\mathfrak{A} \in \mathsf{CA}_\alpha$ and $\mathfrak{A} \subseteq \mathsf{Nr}_\alpha\mathfrak{B}$, with $\mathfrak{B} \in \mathsf{CA}_\beta$ $(\beta > \alpha)$, then we say that $\mathfrak{A}$ *neatly embeds* in $\mathfrak{B}$, and that $\mathfrak{B}$ is a *β–dilation of* $\mathfrak{A}$, or simply a *dilation* of $\mathfrak{A}$ if β is clear from context. For $\mathsf{K} \subseteq \mathsf{CA}_\beta$, and $\alpha < \beta$, $\mathsf{Nr}_\alpha\mathsf{K} = \{\mathsf{Nr}_\alpha\mathfrak{B} : \mathfrak{B} \in \mathsf{K}\} \subseteq \mathsf{CA}_\alpha$. One can show that for any ordinal α, $\mathfrak{A} \in \mathsf{RCA}_\alpha \iff \mathfrak{A} \in \mathbf{S}\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+\omega}$, cf. [12, Theorem 2.6.35]. The last equivalence is Henkin's celebrated neat embedding theorem. For $2 < n < \omega$, what Monk proved is that for any $k \in \omega$, there is an algebra $\mathfrak{A}_k \in \mathbf{S}\mathsf{Nr}_{\mathbf{n}}\mathsf{CA}_{\mathbf{n+k}} \sim \mathsf{RCA}_{\mathbf{n}}$, such that the ultraproduct $\Pi_{k/U}\mathfrak{A}_k/U \in \mathsf{RCA}_n$ for any non–principal ultrafilter on U. This implies that RCA_n is not finitely axiomatizable. If the variety of representable cylindric algebras of dimension at least three had turned out to be axiomatized by a finite schema, algebraic logic would have evolved along a significantly different path than it did in the past fifty years, or so. This would have undoubtedly marked the end of the abstract class CA_α (α an ordinal) as a separate subject of research; after all why bother about abstract algebras, if a few nice extra axioms can lead us from those to concrete algebras consisting of genuine relations, with set theoretic operations uniformly defined over these relations. However, due to Monk's non–finitizability result, together with its improvements by various algebraic logicians (from Andréka to Venema) CA_α was here to stay and its "infinite distance' from RCA_α, when $\alpha > 2$, became an important central research topic. *Monk's non-finite axiomatizability result marked the end of an era and the beginning of a new one.*

Monk's seminal result proved in 1969 [22], showing that the class of representable cylindric algebras of any dimension > 2 is not finitely axiomatizable, had a shattering effect on algebraic logic, in many respects. In fact, it changed the history of the development of the subject, and inspired immensely fruitful research, that involved dozens of publications due to many (algebraic) logicians, starting from Andréka [1] all the way to Venema [6]. The conclusions drawn from this result, were that either the extra non–Boolean basic operations of cylindrifiers and diagonal elements were, due to possibly a historical accident, not properly chosen, or that the notion of representability was inappropriate; for sure it was concrete enough, but perhaps this is precisely the reason; it is far *too concrete.* Research following both paths, either by changing the signature or/and altering the notion of concrete representability have been pursued ever since, with amazing success. Indeed there are two conflicting but complementary facets of such already extensive research referred to in the literature, as 'attacking the representation problem'. One is to sharpen Monk's result proving *negative* (non–finite axiomatizability) results, the other is try to avoid it, proving *positive* results. In the first path one delves deeply in investigating the complexity of potential axiomatizations for existing varieties of representable algebras. The second path involves sidestepping such results by guarding.

3 Part 2: Proofs for the classical case for finite variable fragments

3.1 Connection between non atomicity and Gödel's weak incompleteness

Here we investigate *Gödel's incompleteness* for the finite variable fragments of first order logic in a sense to be made precise in a while. We follow the notation of [12]. In particular, Λ_n denotes a first order language with n variables, $Fm_r^{\Lambda_n}$

denotes the set of all restricted Λ_n formulas and $\vdash_{r,m}$ denotes the proof system in [11, sec 4.3] where m variables are allowed in proofs. To formulate our results, we need a few preparations. We start with a definition:

Definition 3.1. Let $n \in \omega$. Let $n \leq m \leq \omega$. Let Λ_n be a language. We call a formula $\phi \in Fm_r^{\Lambda_n}$ m-consistent if not $\vdash_{r,m} \neg\phi$. (Here m variables are used in the proof.) Let $Fm_r^{\Lambda_n,0}$ be the set of all sentences in $Fm_r^{\Lambda_n}$, i.e.

$$Fm_r^{\Lambda_n,0} = \{\phi \in Fm_r^{\Lambda_n} : \phi \text{ has no free variables }\}.$$

(i) Let $T \subseteq Fm_r^{\Lambda_n,0}$. We say that T is *m-complete* if $(\forall\psi \in Fm_r^{\Lambda_n,0})[T \vdash_{r,m} \psi$ iff not $T \vdash_{r,m} \neg\psi\}$.

(ii) We say that Λ_n has *m-Gödel's incompleteness*, or briefly $m - G.I.$, if there is an m- consistent formula $\phi \in Fm_r^{\Lambda_n}$ such that ϕ cannot be extended to an m-complete recursive theory T, i.e. there is no theory $T \subseteq Fm_r^{\Lambda_n}$ such that $\phi \in T$, T is m-complete and recursive.

Let $n \leq m \leq \omega$. Let Λ_n be a language. We define formula algebras denoted by ${}_{p,m}\mathfrak{Fm}_r^{\Lambda_n}$ and ${}_{p,m}\mathfrak{Fm}_r^{\Lambda_n,0}$. The subscript p indicates that the algebra in question is defined via a provability congruence relation. The subscript m, in turn, indicates that the algebra in question is defined via provability resorting to m variables. In more detail, let

$$\mathfrak{Fm}_r^{\Lambda_n} = \langle Fm_r^{\Lambda_n}, \vee, \wedge, \neg, \exists v_i, v_i = v_j\rangle_{i,j\in n},$$

The algebra $\mathfrak{Fm}_r^{\Lambda_n}$ is the absolutely free algebra of CA_n type [12]. Now let

$$\mathfrak{Fm}_r^{\Lambda_n,0} = \langle Fm_r^{\Lambda_n,0}, \vee, \wedge, \neg\rangle.$$

Note that $\mathfrak{Fm}_r^{\Lambda_n,0}$ is the Boolean reduct of $\mathfrak{Fm}_r^{\Lambda_n}$. Let $n \leq m \leq \omega$. Then

$${}_{m,p}\cong^{\Lambda_n} = \{(\psi, \phi) \in {}^2Fm_r^{\Lambda_n} : \vdash_{r,m} \psi \longleftrightarrow \phi\}.$$

It is straightforward to check that ${}_{m,p}\cong^{\Lambda_n}$ is a congruence relation on the absolutely free algebra $\mathfrak{Fm}_r^{\Lambda_n}$. When n is clear from context, we sometimes write ${}_{m,p}\cong$, or even simply $\cong_m$. Let

$${}_{m,p}\cong^{\Lambda_0} =: {}_{m,p}\cong^{\Lambda_n} \cap {}^2Fm_r^{\Lambda_n,0}.$$

In other words, ${}_{m,p}\cong^{\Lambda_0}$ is the restriction of the congruence relation ${}_{m,p}\cong^{\Lambda_n}$ to the set of sentences, which in turn is a congruence relation on the algebra $\mathfrak{Fm}_r^{\Lambda_n,0}$ of Boolean type. Now set

$${}_{m,p}\mathfrak{Fm}_r^{\Lambda_n} = \mathfrak{Fm}_r^{\Lambda,n}/{}_{m,p}\cong^{\Lambda_n},\ \ {}_{m,p}\mathfrak{Fm}_r^{\Lambda_n,0} = \mathfrak{Fm}_r^{\Lambda_n,0}/{}_{m,p}\cong^{\Lambda_0}\ .$$

${}_{m,p}\mathfrak{Fm}_r^{\Lambda_n}$ is the generic example of a cylindric algebra of dimension n. ${}_{m,p}\mathfrak{Fm}_r^{\Lambda_n,0}$, on the other hand, is a Boolean algebra. We note that for each $\omega > m \geq n$, ${}_{m,p}\mathfrak{Fm}_r^{\Lambda_n,0}$ is a syntactic version (or $\vdash_{r,m}$ version) of the usual Lindenbuam Tarski algebra of Λ. By Gödel's completeness Theorem the latter coincides with ${}_{\omega,p}\mathfrak{Fm}_r^{\Lambda_n,0}$. Also for any $3 \leq n < m < \omega$, if Λ contains at least one binary relation symbol, then the algebras ${}_{n,p}\mathfrak{Fm}_r^{\Lambda_n,0}$, and ${}_{m,p}\mathfrak{Fm}_r^{\Lambda_n,0}$ are distinct. We recall that an atom in a Boolean algebra is a minimal non-zero element. An algebra having a Boolean reduct is atomic if below every non-zero element there is an atom. In our next Lemma we make the connection between atomicity of formula algebras and Gödel's incompleteness explicit:

Theorem 3.2. *Let $n \leq m$. Let Λ_n be a language. If Λ_n has $m - G.I$, then the algebras ${}_{m,p}\mathfrak{Fm}^{\Lambda_n,0}$ and ${}_{m,p}\mathfrak{Fm}^{\Lambda_n}$ are not atomic.*

Proof. We first show that the Boolean algebra of sentences is not atomic. For any formula ϕ, let $\bar{\phi}$ denote its universal closure, i.e. $\bar{\phi}$ is $\forall v_0, \ldots v_n\phi$ where the free variables of ϕ are among $v_0, \ldots, v_n$. Assume that Λ_n has $m - G.I.$ Let ϕ be an incompletable formula. That is ϕ is an m-consistent Λ_n formula that cannot be extended to an m-complete and decidable theory. We write $\cong_m$ for the provability relation using m variables. We show that there is no atom in the Boolean algebra ${}_{m,p}\mathfrak{Fm}_r^{\Lambda_n,0}$ below $\bar{\phi}/\cong_m$. Note that because ϕ is m-consistent, it follows that $\bar{\phi}/\cong_m$ is non-zero. Now, assume to the contrary that there is such an atom $\tau/\cong_m$, for some sentence τ. This means that $\tau/\cong_m \cdot\bar{\phi}/\cong_m = \tau/\cong_m$ i.e. that $(\tau \wedge \bar{\phi})/\cong_m = \tau/\cong_m$. Then it follows that $\vdash_m (\tau \wedge \phi) \to \phi$, i.e. $\vdash_m \tau \to \phi$. Let $T = \{\tau, \phi\}$, and let $Consq(T) = \{\psi \in Fm_r^{\Lambda_n,0} : T \vdash_m \psi\}$. $Consq(T)$ is short for the consequences of T. We show that T is m-complete and that $Consq(T)$ is decidable. Let ψ be an arbitrary sentence in ${}_{m,p}Fm_r^{\Lambda,0}$. Then either $\tau/\cong_m \leq \psi/\cong_m$ or $\tau/\cong_m \leq \neg\psi/\cong_m$ because $\tau/\cong$ is an atom. Thus $T \vdash_{n,m} \psi$ or $T \vdash_{n,m} \neg\psi$. Here, it is the *exclusive or*, i.e., the two cases cannot occur together. This is because $T \vdash_m \psi$ iff $\vdash_m \tau \to \psi$. So if $\vdash_m \tau \to \psi$ and $\vdash_m \tau \to \neg\psi$ then $\vdash_m \tau \to \bot$. This contradicts that $\tau/\cong_m$ is an atom. Clearly $ConsqT$ is recursively enumerable. By m-completeness of T we have ${}_{m,p}Fm_r^{\Lambda_n,0} \smallsetminus Consq(T) = \{\neg\psi : \psi \in Consq(T)\}$, hence the complement of $ConsqT$ is recursively enumerable as well, hence T is decidable. Here we are using the trivial fact that ${}_{m,p}\mathfrak{Fm}^{\Lambda_n,0}$ is decidable. This contradiction proves that ${}_{p,m}\mathfrak{Fm}^{\Lambda_n,0}$ is not atomic. Now we prove that the formula algebra is not atomic. The idea is that atoms in the algebra of formula can be "pulled back" to atoms in the algebra of sentences. In more detail, again seeking a contradiction assume that the algebra of formulas is atomic. Let b be a non-zero element in the algebra of sentences. Then b is a non-zero element in the bigger algebra of formulas. The latter is atomic, so there exists an atom $\tau/\cong_m$ below b. But it is straightforward to check that if $\psi/\cong_m$ is an atom in ${}_{m,p}\mathfrak{Fm}^{\Lambda_n}$, then $\bar{\psi}/\cong$ is an atom in ${}_{m,p}\mathfrak{Fm}^{\Lambda_n,0}$. So $\bar{\tau}/\cong_m$ is an atom in the algebra of sentences below b. Since b was an arbitrary non-zero element, we get that the algebra of sentences is atomic. Contradiction, proving that the algebra of formulas is also atomic. $\square$

Gödel's Incompleteness Theorem is a sign of strength (of expression) rather than weakness. The next Theorem, due to Németi, shows that in the presence of at least 3 variables, first order logic, is still strong enough to render a Gödel's incompleteness result, and indeed we have:

Theorem 3.3. *(Gödel-Németi) Let Λ be a first order language with at least one binary relation symbol. Then the following hold:*

- *(i) Λ_3 has $3-G.I.$ Therefore, ${}_{3,p}\mathfrak{Fm}^{\Lambda_3}$ is not atomic.*
- *(ii) Let $3 < n \leq m \leq \omega$. Then Λ_n has $m-G.I.$ Therefore, ${}_{m,p}\mathfrak{Fm}_r^{\Lambda_n}$ is not atomic.*

Proof. Gödel proved Gödel's incompleteness for $m = n = \omega$. Using Gödel's result, or rather generalizing it, together with the pairing technique of Tarski, Németi [37] proves (i) and the rest of the cases in (ii). $\square$

We now state algebraic corollaries to Theorem 4.3. But first some definitions are recalled from [11].

Definition 3.4. [11, 2.5.31] Let δ be a cardinal. Let α be an ordinal. Let ${}_\alpha\mathfrak{Fr}_\delta$ be the absolutely free algebra on δ generators and of type CA_α. For an algebra $\mathfrak{A}$, we write $R \in Con\ A$ if R is a congruence relation on $\mathfrak{A}$. Let $\rho \in {}^\delta\wp(\alpha)$. Let L be a class having the same similarity type as CA_α. SL denotes the class of all subalgebras of members of L. Let

$$Cr_\delta^{(\rho)}L = \bigcap\{R : R \in Co_\alpha\mathfrak{Fr}_\delta, {}_\alpha\mathfrak{Fr}_\delta/R \in SL, c_k^{\alpha\mathfrak{Fr}_\delta}\eta/R = \eta/R \text{ for each}$$

$$\eta < \delta \text{ and each } k \in \alpha \smallsetminus \rho\eta\}$$

and

$$\mathfrak{Fr}_\delta^\rho L = {}_\alpha\mathfrak{Fr}_\beta/Cr_\delta^{(\rho)}L.$$

The ordinal α does not appear in $Cr_\delta^{(\rho)}L$ and $\mathfrak{Fr}_\delta^{(\rho)}L$ though it is involved in their definition. However, α will be clear from context so that no confusion is likely to ensue. The algebra $\mathfrak{Fr}_\delta^{(\rho)}L$ is referred to [11] as a dimension restricted free algebra over K with β generators. Also $\mathfrak{Fr}_\delta^{(\rho)}L$ is said to be dimension restricted by the function ρ, or simply, ρ-dimension-restricted.

Definition 3.5. Let δ be a cardinal. Assume that $L \subseteq \mathsf{CA}_\alpha$, $x = \langle x_\eta : \eta < \delta\rangle \in {}^\delta A$ and $\rho \in {}^\delta\wp(\alpha)$. Then we say that the sequence x L-freely generates $\mathfrak{A}$ under the dimension restricting function ρ, if the following two conditions are satisfied:

(i) $\mathfrak{A}$ is generated by Rngx, and $\Delta x_\eta \subseteq \rho(\eta)$ for every $\eta < \delta$.

(ii) Whenever $\mathfrak{B} \in L$, $y = \langle y_\eta : \eta < \delta\rangle \in {}^\delta B$ and $\Delta y_\eta \subseteq \rho(\eta)$ for every $\eta < \delta$, there is a homomorphism h from $\mathfrak{A}$ to $\mathfrak{B}$ such that $h \circ x = y$.

For an algebra $\mathfrak{A}$ and $X \subseteq A$, we write, following [11], $\mathfrak{Sg}^{\mathfrak{A}}X$, or even simply $\mathfrak{Sg}X$, for the subalgebra of $\mathfrak{A}$ generated by X. We have the following which almost follow from the definitions.

Theorem 3.6. *Let $L \subseteq \mathsf{CA}_\alpha$. Let δ be a cardinal. Let $\rho \in {}^\delta\wp(\alpha)$. Then the sequence $\langle \eta/Cr_\delta^{(\rho)}L : \eta < \delta\rangle$ L-freely generates $\mathfrak{Fr}_\delta^\rho L$ under the dimension restricting function ρ.*

Proof. [12, 2.5.37] and [12, 2.6.45]. □

Theorem 3.7. *If $\Lambda_n = (n, R, \rho)$ is any language with $\mathsf{Dom}R = \mathsf{Dom}\rho = \beta$ and $n \leq m$, then ${}_{m,p}\mathfrak{Fm}_r^{\Lambda_n} \cong \mathfrak{Fr}_\beta^\rho \mathbf{S}\mathfrak{Nr}_n\mathsf{CA}_m$, where $\mathbf{S}$ stands for the operation of forming subalgebars.*

Proof. Let $n < \omega$. For brevity, let $\mathfrak{B} = {}_{m,p}\mathfrak{Fm}_r^{\Lambda_n}$ Then $\mathfrak{B} \in \mathsf{CA}_m$. Let $x = \langle R_i/\cong_m : i < \beta\rangle$. Consider any $\mathfrak{A} \in \mathfrak{Nr}_n\mathsf{CA}_m$ and $y \in {}^\beta A$ such that $\Delta y_i \subseteq \rho(i)$ for $i < \beta$. Let $\mathfrak{C} \in \mathsf{CA}_m$ such that $\mathfrak{A} = \mathfrak{Nr}_n\mathfrak{C}$. Clearly $y \in {}^nC$ and $\Delta y_i \subseteq n$ for all $i < \beta$. Now x CA_n freely generates $\mathfrak{B}$ under the dimension restricting function ρ, and so there is an $h \in Hom(\mathfrak{B}, \mathfrak{C})$ such that $h \circ x = y$. Therefore $h : \mathfrak{Rd}_n\mathfrak{B} \to \mathfrak{Rd}_n\mathfrak{C}$ isd a homorphism. . But $\mathfrak{Sg}^{\mathfrak{Rd}_n\mathfrak{B}}\mathsf{Rng}x \subseteq \mathfrak{Rd}_n\mathfrak{B}$ and $h(\mathfrak{Sg}^{\mathfrak{Rd}_n\mathfrak{B}}\mathsf{Rng}x) = \mathfrak{Sg}^{\mathfrak{Rd}_n\mathfrak{C}}(h\mathsf{Rng}x)$ and so $h : \mathfrak{Sg}^{\mathfrak{Rd}_n\mathfrak{B}}\mathsf{Rng}x, \mathfrak{Sg}^{\mathfrak{Rd}_n\mathfrak{C}}h\mathsf{Rng}x$. is a homomorphism But $\mathsf{Rng}x \subseteq \mathfrak{Nr}_n\mathfrak{B}$ and $h(\mathsf{Rng}x) = \mathsf{Rng}y \subseteq \mathfrak{Nr}_n\mathfrak{C}$. We readily obtain that $h : \mathfrak{Sg}^{\mathfrak{Nr}_n\mathfrak{B}}\mathsf{Rng}x \to \mathfrak{A}$ is a homorphism. But $\mathfrak{Sg}^{\mathfrak{Nr}_n\mathfrak{B}}\mathsf{Rng}x \cong {}_{m,p}\mathfrak{Fm}_r^{\Lambda_n}$, hence the desired conclusion. □

Corollary 3.8. *(i) Let $\omega \geq m \geq 3$. Let β be a non-zero cardinal $< \omega$ and $\rho : \beta \to \wp(3)$ such that $\rho(i) \geq 2$ for some $i \in \beta$. Then $\mathfrak{Fr}_\beta^\rho S\mathsf{Nr}_3\mathsf{CA}_m$ is not atomic. In particular, $\mathfrak{Fr}_\beta^\rho\mathsf{CA}_3$ and $\mathfrak{Fr}_\beta^\rho\mathsf{RCA}_3$ are not atomic.*

(ii) Let $m \geq n > 3$. Let β be a non-zero cardinal $< \omega$ and let $\rho : \beta \to \wp(n)$ such that $\rho(i) \geq 2$ for some $i \in \beta$. Then $\mathfrak{Fr}_\beta^\rho \mathbf{S}\mathfrak{Nr}_\mathbf{n}\mathsf{CA}_\mathbf{m}$ is not atomic. In particular, $\mathfrak{Fr}_\beta\mathsf{CA}_4$ and $\mathfrak{Fr}_\beta\mathsf{RCA}_4$ are not atomic.

The non-atomicity of $\mathfrak{Fr}_\beta\mathsf{CA}_3$, $\mathfrak{Fr}_\beta\mathsf{RCA}_3$ solves problem 4.14 in [12].

Because our proof goes via the route of relation algebras, we shall also prove:

Theorem 3.9. *Let* RA *and* RRA *stand for the classes of relation algebras and representable relation algebras, and lket β be fiuite non-zero cardinal. . Then* $\mathfrak{Fr}_\beta$RA *and* $\mathfrak{Fr}_\beta RRA$*. In fact for any class* K *such that* RRA $\subseteq$ K $\subseteq$ RA $\mathfrak{Fr}_\beta$K *is not atomic.*

3.2 Detailed proofs of non atomicity of free cylindric and relation algebras

This part is essentially due to Németi [37] building on work of Tarski and Givant [35] on coding ZFC in the calculus of relations. Let $2 \leq n \leq \omega$. Λ_n denotes the language with equality having (for the time being) countably many relation symbols each of arity $\leq n$ and having $\{v_i : i \in n\}$ as a set of distinct variables. In what follows, we shall write x, y, z for meta variables denoting v_0, v_1, v_2, respectively. That is x, y, z are distinct and any of x, y, z can be any of v_0, v_1, v_2. For brevity, we write Fm_n for the set of all restricted Λ_n formulas. When we write $\phi(x, y)$, we shall understand a formula whose free variables are among x and y. Assume that $\phi(x, y)$ is such a restricted formula. Because we are allowed only restricted formulas, throughout we use the following *substitution* convention (S):

$$\phi(x,z) = \exists y(y = z \wedge \phi(x,y)),$$

$$\phi(y,z) = \exists x(x = y \wedge \phi(x,z)),$$

$$\phi(y,x) = \exists z(z = x \wedge \phi(y,z)),$$

$$\phi(z,x) = \exists y(y = z \wedge \phi(y,x)),$$

$$\phi(z,y) = \exists x(x = z \wedge \phi(x,y)),$$

$$\phi(x,x) = \exists y(y = x \wedge \phi(x,y)),$$

$$\phi(y,y) = \exists x(x = y \wedge \phi(x,y)),$$

$$\phi(z,z) = \exists x(x = z \wedge \phi(x,x)),$$

The word substitution here comes from the fact that $\phi(x, z)$ for example is the formula that is semantically equivalent to the formula obtained from $\phi(x, y)$ by replacing the free occurrences of y by z such that the substitution is free, i.e z is free in the new formula. For $k < n$, we let

$$Fm_n^k = \{\phi \in Fm_n : \text{ all the free variables of } \phi \text{ are among the first } k\}.$$

Let $p_0(x, y)$, $p_1(x, y) \in Fm_3^2$ be arbitrary. Define $\pi \in Fm_3^0$ as follows:

$$\pi = (\forall x)(\forall y)(\forall z)[(p_0(x,y) \wedge p_0(x,z) \implies y = z) \wedge$$

$$p_1(x,y) \wedge p_1(x,z) \implies y = z) \wedge$$
$$\exists z(p_0(z,x) \wedge p_1(z,y)].$$

π expresses the fact that p_0 and p_1 form a pair of *quasiprojections*, that is to say in a model M say of π, p_0 and p_1 are functions and for any element $a, b \in \mathsf{M}$, there is a c such that p_0 and p_1 map c to a and b, respectively. We can think of c as representing the ordered pair (a, b) and p_0 and p_1 are the functions that project the ordered pair onto its first and second coordinates. Note that writing π without the substitution convention, i.e. writing π as a restricted formula would be much longer.

In what follows, $\models$ denotes the semantical consequence relation. The following Lemma, due to Tarski, is known since it states a basic property of Tarski's pairing functions, namely we can code up, or represent, any sequence of variables in terms of a single variable, thus effectively reducing the number of variables to one. In more detail, we have:

Lemma 3.10. *Assume that Λ_3 has only one binary relation symbol ϵ. Let $p_0(x,y)$ and $p_1(x,y)$ be in Fm_3^2. Let π be as described above. Then there is a recursive function $f : Fm_\omega^2 \to Fm_3$ such that $(i) - (iii)$ below hold for every $\phi \in Fm_\omega^2$.*

- *(i)* $\pi \models \phi \longleftrightarrow f\phi$
- *(ii)* $f(\neg\phi) = \neg f(\phi)$,
- *(iii)* $f\phi \in Fm_3^j$ *whenever* $\phi \in Fm_\omega^j$ *for every* $j \leq 2$.

Proof. Let Fm_3' denote the language Fm_3 enriched with two unary partial function symbols p_0, p_1 and such that we consider *all* formulas, and not just restricted ones. In other words, Fm_3' consists of all first order formulas built up from one binary relation symbol ϵ, two unary function symbols p_0 and p_1 and using x, y, z as variables. Here we are using the same symbol p_i, $i \in 2$, for two different things. What follows this symbol will clarify matters. When we write $p_0(x)$ we understand p_0 as a unary function. When we write $p_0(x,y)$ we understand the formula $p_0(x,y)$ containing at most x and y as free variables. Note that $p_0(x,y)$ is a formula in Fm_3^2 to be later specified, while p_0 is a new function symbol. Let

$$\pi_p = \pi \wedge \{p_i(x,y) \longleftrightarrow p_i(x) = y : i \in 2\}.$$

The formula just defined justifies our use of p_i in two different contexts. In what follows we use the validity relation $\models_p$ to denote that we use the logic where p_0 and p_1 denote only *partial* functions. For example:

$$\pi_p \models (\forall xy)\exists z(p_0 z = x \wedge p_1 z = y).$$

We construct the desired function f in two steps. First, we show the existence of $f' : Fm^2_\omega \to Fm'_3$ with the required properties (but using $\pi_p \models_p$) instead of $\pi \models$. Then we obtain our desired f from f'.

Towards the end of constructing f', we recall that there is a standard algorithm of obtaining a prenex normal form $pr(\psi)$ of every first order formula, and in particular, of any $\psi \in Fm^2_\omega$. $pr(\psi)$ is a formula of the form $Q\phi(x,y)$ where Q is a sequence of existential quantifiers and negation symbols $\neg$, $\phi(x,y)$ is a quantifier free formula containing only variables occurring in Q together with x, y, each variable occurs only once in Q, and x, y, z do not occur in Q. Also, we have $pr(\neg\psi) = \neg(pr(\psi))$ for every $\psi \in Fm^2_\omega$.

Let $\psi \in Fm^2_\omega$ and $pr(\psi)$ be $Q\phi(x,y)$ with the above properties. Let w be a variable. Then $\phi(x, p_0y, w|p_1y)$ denotes the formula we obtain from $\phi(x,y)$ by replacing y, w with p_0y, p_1y respectively everywhere in $\phi(x,y)$, simultaneously. Assume that Q is $v\exists wQ'$ for some (possibly empty) sequence v of the negation symbol and for some variable w and Q'. Then it is not difficult to check that

$$(1)\pi_p \models_p v\exists wQ'\phi(x,y) \longleftrightarrow v\exists z(p_0z = y \wedge \exists y[y = z \wedge Q'\phi(x, p_0y, w|p_1y)].$$

Now based on (1) above, one can define the first required function $f' : Fm^2_\omega \to Fm'_3$ by an obvious recursion. Next we want to get rid of the function symbols p_0, p_1 and of the formulas that are not restricted. Recall that we have only one relation symbol ϵ that is binary. Let τ, σ be finite sequences of p_0, p_1 and let $i \in 2$. Recall that x, y, z are meta variables for v_0, v_1, v_2. Then it is easy to check that:

$$(2)\ \pi_p \models_p \epsilon(\tau x, \sigma y) \longleftrightarrow \exists z[p_0z = \tau x \wedge p_1z = \sigma y \wedge \epsilon(p_0z, p_1z)],$$

$$(3)\ \pi_p \models \epsilon(p_0z, p_1z) \longleftrightarrow \exists x\exists y[x = p_0z \wedge y = p_1z \wedge \epsilon(x,y)].$$

$$(4)\quad \pi_p \models_p \tau(x) = \sigma(x) \longleftrightarrow \exists z[p_0z = \tau(x) \wedge p_1z = \sigma y \wedge p_0z = p_1z]$$

$$(5)\quad \pi_p \models_p p_ix = \tau y \longleftrightarrow \exists z[z = p_ix \wedge z = \tau y]$$

$$(6)\ \pi_p \models_p x = p_i\tau y \longleftrightarrow \exists z[p_i(z,x) \wedge z = \tau y].$$

Based on $(2) - (6)$ and on convention (S), one can define a recursive function $g : Fm'_3 \to Fm_3$ such that

$$(\forall\phi \in Fm'_3)[\pi_p \models_p \phi \longleftrightarrow g(\phi) \text{ and } g(\neg\phi) = \neg(g\phi)].$$

Noticing that $\pi \models \phi$ if and only if $\pi_p \models_p \phi$ for every $\phi \in Fm_\omega$ completes the proof. $\square$

We now start constructing our incompletable formula. We start working with ordinary first order logic. We construct a so-called *inseparable* sentence, which is incompletable in usual first order logic. Our sentences will be a consequence of Peano arithmetic in a sense. This part is based on the textbook [10] .We start by stating what we mean by inseparable:

Definition 3.11. (i) Let λ be a first order sentence, i.e $\lambda \in Fm_\omega^0$. We call λ inseparable iff there is no set $T \subseteq Fm_\omega^0$ which recursively separates the theorems of λ from the refutable sentences of λ, i.e. iff there is no recursive set $T \subseteq Fm_\omega^0$ such that

$$\{\phi \in Fm_\omega^0 : \lambda \models \phi\} \subseteq T \subseteq \{\phi \in Fm_\omega^0 : \lambda \models \neg\phi\}.$$

$$\pi' = \pi \wedge \forall xyzv_3[p_0(z,x) \wedge p_1(z,y) \wedge p_0(v_3,x) \wedge p_1(v_3,y)) \to z = v_3].$$

Then $\pi' \in Fm_4^0$.

Lemma 3.12. *(Essentially Tarski) Let Λ be a language with only one binary relation symbol ϵ. Then there exists an inseparable $\lambda \in Fm_\omega^0$ and $p_i(x,y) \in Fm_3^2$ such that λ is semantically consistent with π', i.e. $\lambda \wedge \pi'$ has a model.*

Proof. Let A_E stand for the finite set of axioms of arithmetic in [10] p. 203. Then A_E is inseparable. Our inseparable λ will be A_E translated to set theory (and relativized to finite ordinals), while $p_i(x,y)(i \in 2)$ will be formulas in set theory expressing the usual intended meaning. Then λ will be inseparable and the model $\mathcal{H} = (H, \epsilon)$ will be a model of $\lambda \wedge \pi'$. We first define $p_i(x,y)$ for $i \in 2$. For convenience, we write $x\epsilon y$ instead of $\epsilon(x,y)$.

$$x = \{y\} =: y\epsilon x \wedge (\forall z)(z\epsilon x \implies z = y)$$

$$\{x\}\epsilon y =: \exists z(z = \{x\} \wedge z\epsilon y)$$

$$x = \{\{y\}\} =: \exists z(z = \{y\} \wedge x = \{z\})$$

$$x\epsilon \cup y := \exists z(x\epsilon z \wedge z\epsilon y)$$

$$pair(x) =: \exists y[\{y\}\epsilon x \wedge (\forall z)(\{z\}\epsilon x \to z = y)] \wedge \forall zy[(z\epsilon \cup x \wedge \{z\} \notin x \wedge$$
$$y\epsilon \cup x \wedge \{y\} \notin x \to z = y] \wedge \forall z\epsilon x \exists y(y\epsilon z).$$

Now we define the pairing functions:

$$p_0(x,y) =: pair(x) \wedge \{y\}\epsilon x$$

$$p_1(x,y) =: pair(x) \wedge [x = \{\{y\}\} \vee (\{y\} \notin x \wedge y\epsilon \cup x)].$$

$p_0(x,y)$ and $p_1(x,y)$ are defined. We now formulate the desired λ. We will be more sketchy here:

$x\epsilon Ord$ =: " x is an ordinal, i.e. x is transitive and ϵ is a total ordering on x,

$x\epsilon Ford$ =: $x\epsilon Ord \wedge$ "every element of x is a successor ordinal "
i.e. x is a finite ordinal .

$$x = 0 =: \text{“}x \text{ has no element ”}$$

$$sx = z =: z = x \cup \{x\},$$

$$x \leq y =: x \subseteq y,$$

$$x < y =: x \leq y \wedge x \neq y,$$

$$x + y = z =: \exists v(z = x \cup v \wedge x \cap v = 0 \wedge$$

$$\text{“there exists a bijection between } v \text{ and } y\text{”})$$

$$x \cdot y = z =: \text{ “there is a bijection between } z \text{ and } x \times y\text{”}$$

$$x\underline{exp}y = z : \text{ there is a bijection between } z \text{ and the set of all functions from } y \text{ to } x\text{”}$$

Now λ' is the formula saying that: $0, s, +, \cdot, \underline{exp}$ are functions of arities $0, 1, 2, 2, 2$ on $Ford$ and

$$(\forall xy \epsilon Ford)[sx \neq 0 \wedge sx = sy \to x = y) \wedge (x < sy \longleftrightarrow x \leq y) \wedge$$

$$\neg(x < 0) \wedge (x < y \vee x = y \vee y < x) \wedge (x + 0 = x) \wedge (x + sy = s(x + y)) \wedge (x.0 = 0)$$

$$\wedge (x \cdot sy = x \cdot y + x) \wedge (x\underline{exp}0 = s0) \wedge (x\underline{exp}sy = x\underline{exp}y \cdot x)].$$

Now the existence of the desired incompletable λ readily follows: $\lambda \in Fm^0_\omega$ is defined to be the restricted form of the above λ'. □

We have an incompletable sentence λ. But we are not there yet. We need to translate λ into L_4, the first order logic restricted to the first 4 variables. We use Tarski's techniques of going from first order logic with infinitely many variables to L_4 via relation algebras: For this purpose, in what follows we make a detour in the theory of relation algebras.

3.3 Relation Algebras

Let RA, Rs denote the classes of relation algebras and all relation set algebras, respectively. **SimRA** denotes the class of all algebras similar to RA's. Then for e.g. $\mathsf{RA} \subseteq$ **SimRA**. Let X be a set. Then $\mathfrak{Fr}_X$**SimRA** is the set of all relation algebraic terms written up from the elements of X as variable symbols or the basolutely free algebra having the same signature as RA. Then Fr_X**SimRA** is the subuniverse of the free **SimRA** algebra generated by X in accordance with the notation of [11]. We may write RAT_X or just RAT instead of $\mathfrak{Fr}_X$**SimRA**. Thus RAT_X is the smallest set such that (i)-(iii) below hold:

(i) $R \in \mathsf{RAT}_X$ for every $R \in X$,

(ii) $Id, 0, 1 \in \mathsf{RAT}_X$

(iii) $\smile\tau$, $\tau;\sigma, \neg\tau, \tau\cdots\sigma, \tau+\sigma \in \mathsf{RAT}_X$, if $\tau,\sigma \in \mathsf{RAT}_X$.

(Here Id stands for the identity relation and $\smile$ and ; stand for inversion and composition of relations.)

Let $X \in \mathcal{A} \in \mathsf{RA}$ and $\tau \in \mathsf{RAT}_1$, where 1 is the Von Neumann ordinal $\{0\}$. Then $\tau^{\mathcal{A}}(X)$ denotes the element $h(\tau) \in \mathfrak{A}$ where $h : \mathfrak{Fr}_1\mathbf{SimRA} \to \mathfrak{A}$ is the homomorphism taking the free generator of $\mathfrak{Fr}_1\mathbf{SimRA}$ to X. If $\mathfrak{A} \in \mathsf{Rs}$, the class of relation set algebras, then the inverse of $\mathfrak{A}$ is of the form $\wp({}^2U)$ such that U is a set, called the *base* of $\mathfrak{A}$, and denoted by $\mathsf{base}(\mathfrak{A})$.

Our next Lemma, is basically the same as Lemma 5.3.12 of [11]. It says that every element of Fm_3^2 can be expressed with a relation algebraic term.

Lemma 3.13. *There is a recursive function* $r : \mathfrak{Fm}_3^2 \to \mathsf{RAT}$ *such that:*

(i) $(r\phi)^{\mathcal{A}}(X) = \{a \in 1^{\mathfrak{A}} : (\mathsf{base}(\mathfrak{A}), X) \models \phi[a]\}$, *for every* $X \in \mathfrak{A} \in \mathsf{Rs}$.

(ii) $r(\neg\phi) = -r(\phi)$.

Proof. Let RAT denote the set of all relation algebraic terms over the single variable symbol (or generator) R. That is $\mathsf{RAT} = \mathsf{RAT}_\mathsf{R} = \mathfrak{Fr}_\mathsf{R}\mathbf{SimRA}$. For $\phi \in \mathfrak{Fm}_3^2$ and $i,j \in 3$, let $\mathsf{s}_j^i\phi = \exists v_i(v_i = v_j \wedge \phi)$. We define a function $t : \mathsf{RAT} \to \mathfrak{Fm}_3^2$ as follows:

$$t(R) =: \epsilon(x,y), t(Id) =: x = y, t(1) =: \top, t(0) =: \bot.$$

Let $\tau,\sigma \in RAT$. Then $t(\tau^\vee) = \mathsf{s}_0^2\mathsf{s}_1^0\mathsf{s}_2^1 t(\tau)$, $t(\tau;\sigma) = \exists v_2(s_2^1 t(\tau) \wedge s_2^0 t(\sigma))$, $t(-\tau) = \neg t(\tau)$, $t(\tau.\sigma) = t(\tau) \wedge t(\sigma)$, $t(\tau+\sigma) = t(\tau) \vee t(\sigma)$. It is easy to check that $\tau^{\mathfrak{A}}X = \{a \in 1^{\mathfrak{A}} : (base(\mathfrak{A}), X) \models t(\tau)[a]\}$, for every $X \in \mathfrak{A} \in \mathsf{Rs}$ and $\tau \in \mathsf{RAT}$. Let

$$\mathcal{R} = \{X \subseteq {}^3\mathsf{RAT} : |X| < \omega\}.$$

First we define a recursive function $\rho : \mathfrak{Fm}_3 \to \mathcal{R}$ with the following properties:

$$(i)' \models \phi \longleftrightarrow \bigwedge\{t(r_0) \wedge \mathsf{s}_2^1 t(r_1) \wedge \mathsf{s}_2^0 t(r_2) : r \in \rho(\phi)\}$$

$(ii)'$ If $\phi \in Fm_3^2$ then $\rho(\phi) = \{(\tau, 1, 1)\}$ for some τ in RAT.

$(iii)'$ $\rho(\neg\phi) = \{(-r_0, 1, 1)\}$ if $\phi \in \mathfrak{Fm}_3^2$ and $r \in \rho(\phi)$.

We may well assume that the elements of $\mathfrak{Fm}_3^2$ are built up from $\epsilon(v_0, v_1)$, $v_i = v_j$ by $\vee$, $\neg$ $\exists v_i$ $(i,j \in 3)$. We define ρ by (1) – (8) below:

(1) $\rho(\epsilon(v_0, v_1)) = \{(R, 1, 1)\}$

(2) $\rho(v_i = v_i) = \{(1,1,1)\}$, for $i \in 3$.

(3) $\rho(v_0 = v_1) = \rho(v_1 = v_0) = \{(Id, 1, 1)\}$
$\rho(v_0 = v_2) = \rho(v_2 = v_0) = \{1, Id, 1)\}$
$\rho(v_1 = v_2) = \rho(v_2 = v_1) = \{(1, 1, Id)\}$
Let ϕ, $\psi \in Fm_3$. Then

(4) $\rho(\psi \vee \phi) = \rho\phi \cup \rho\psi$ if $\phi \vee \psi \notin Fm_3^2$
$=\{(\sum\{r_0 : r \in \rho\phi \cup \rho\psi\}, 1, 1)\}$, if $\phi, \psi \in Fm_3^2$.

(5) $\rho(\neg\phi) = \{(\prod\{-r_0 : r \in H_0\}, \prod\{-r_1 : r \in H_1\}, \prod\{-r_2 : r \in H_2\})$
$:H_0 \cup H_1 \cup H_2 = \rho\phi, H_i \cap H_j = \emptyset, i < j < 3\}$ if $\phi \notin Fm_3^2$,
$= \{(-r_0, 1, 1)\}$: if $\phi \in Fm_3^2$ and $\rho(\phi) = \{r\}$.

(6) $\rho(\exists v_2\phi) = \{\sum\{r_0.(r_1; r_2) : r \in \rho\phi\}, 1, 1)\}$.

(7) $\rho(\exists v_0\phi) = \{(1, 1, \sum\{r_2 \cdot (r_1^\vee; r_0) : r \in \rho\phi\})\}$ if $\phi \notin Fm_3^2$
$= \{(1; \sum r_2.(r_1^\vee; r_0) : r \in \rho\phi\}, 1, 1)\}$ if $\phi \in Fm_3^2$

(8) $\rho(\exists v_1\phi) = \{(1, \sum\{r_1 \cdot (r_0; r_2^\vee) : r \in \rho\phi\}, 1)\}$ if $\phi \notin Fm_3^2$
$= \{(1; \sum r_1 \cdot (r_0; r_2^\vee) : r \in \rho\phi\}; 1, 1, 1)\}$ if $\phi \in Fm_3^2$

Then $\rho : \mathfrak{Fm}_3^2 \to \mathcal{R}$ is recursive. Also it is not difficult to check that $i)' - (iii)'$ hold. Let $\phi \in Fm_3^2$. Then we define

$$r(\phi) = \tau \text{ where } \rho(\phi) = \{\tau, 1, 1)\}.$$

Then $r : \mathfrak{Fm}_3^2 \to \mathsf{RAT}$ is recursive. Also we have $\models \phi \longleftrightarrow t(r\phi)$ by (i)' and (ii)', hence (i) and (ii) hold. □

Our next Lemma is crucial, showing that inseparable sentences can be coded inside the language of RA in such a way that they cannot be extended to decidable congruences. This technique can be traced back to Tarski and Givant [35]. But first we fix some needed notation:

Notation . *Let $p_0(x, y)$ and $p_1(x, y)$ be the paring functions as defined above. Let r be the recursive function mapping $\mathfrak{Fm}_3^2$ into* RAT. *Now let*

$$p = r(p_0(x, y)), q = r(p_1(x, y))$$

and

$$\pi_{\mathsf{RA}} = (p^\vee; p \to Id) \cdot (q^\vee; q \to Id) \cdot (p^\vee; q).$$

Then $\pi_{\mathsf{RA}} \in$ RAT *since $p_i(x, y) \in Fm_3^2$. f is as defined in Lemma 3.12.*

Now we ready for:

Lemma 3.14. *Let $\lambda \in \mathfrak{Fm}_\omega^0$ be inseparable and let $\eta = (rf\lambda)\cdot\pi_{RA}$. Then there is no decidable proper congruence $R \in \mathsf{Co}(\mathfrak{Fr}_1\mathbf{SimRA})$ such that $\eta \in 1/R$ and $\mathfrak{Fr}_{\mathbf{SimRA}}/R \in \mathsf{RA}$.*

Proof. Assume, seeking a contradiction, that R is such. Define

$$T = \{\phi \in Fm_\omega^0 : rf\phi \in 1/R\}.$$

We show that T recursively separates the theorems of λ from the refutable sentences of λ which contradicts the choice of λ. T is recursive because r, f and R are decidable. Assume now that $\mu \in \mathfrak{Fm}_\omega^0$ is such that $\lambda \models \mu$. We will show that $\mu \in T$. Let $\mathcal{F} = \mathfrak{F}r_1\mathbf{SimRA}/R$. Then $\mathcal{F} \in \mathsf{RA}$, and $\bar{p} = p/r$ $\bar{q} = q/R$ are *pairing functions* in $\mathcal{F}$ by $\pi_{\mathsf{RA}} \in 1/R$. Therefore $\mathcal{F}$ is a representable RA, by Tarski's theorem $\mathsf{QRA} \subseteq \mathsf{RRA}$ cf. [41, 38]. Assume that $\mu \notin T$. This means that $rf\mu \notin 1/R$. By $\mathcal{F} \in \mathsf{RRA}$ and $\mu \in 1/R$, then there are $\mathfrak{A} \in \mathsf{Rs}$ and $Z \in A$ such that $\eta^{\mathfrak{A}}(Z) = 1^{\mathfrak{A}}$ while $(rf\mu)^{\mathfrak{A}}(Z) \neq 1$. By $\eta = (rf\lambda)\cdot\pi_{RA}$, we also have $(rf\lambda)^{\mathfrak{A}}(Z) = 1$ and $\pi_{\mathsf{RA}}^{\mathfrak{A}}(Z) = 1$. Let $U = \mathsf{base}(\mathfrak{A})$ and $\mathsf{M} = (U, Z)$. We then have:

$$(*)\ \ \mathsf{M} \models f\lambda, \text{ and not } \mathsf{M} \models f(\mu),$$

$$(**)\ \ p(Z) = \{(u,v) \in {}^2U : \mathsf{M} \models p_0(u,z)\}$$

$$q(Z) = \{(u,v) \in {}^2U : \mathsf{M} \models p_1(u,z)\}.$$

By $\pi_{\mathsf{RA}}^{\mathfrak{A}}(Z) = 1$ and (**) we then have $\mathsf{M} \models \pi$. By (*), we have $\mathsf{M} \models \lambda$, and M does not model μ, contradicting $\lambda \models \mu$. Thus $\mu \in T$ for every $\mu \in Fm_\omega^0$ for which $\lambda \models \mu$. I.e T contains the theorems of λ. Assume that $\lambda \models \neg\mu$ for some $\mu \in Fm_\omega^0$. We show that $\mu \notin T$. We have $\neg\mu \in T$ by $\lambda \models \neg\mu$ i.e $(rf)(\neg\mu) \in 1/R$. But $rf(\neg\mu) = -rf(\mu)$ hence $rf(\mu) \in 0/R \neq 1/R$, i.e $rf(\mu) \notin 1/R$. This means that $\mu \notin T$. Thus T is disjoint from the refutable sentences of λ. □

We are now ready to prove that if Λ be a first order language with at least one relation symbol of arity $\geq$ 2and $m > 3$, then Λ_3 has $m - G.I.$ We first prove this when Λ has exactly one relation symbol of arity 2. Let us call it $\in$. Fix $m > 4$. Let $p_i(x,y)$, π', λ, r, f and π_{RA} be as above. All of these formulas are built up only of $x \in y$ beside equality. For $\mathfrak{D}$ having the same signature as $\mathsf{CA_l}$ type, with $l \geq 3$, let $\mathsf{Ra}\mathfrak{D}$ be its relation algebra reduct. $\mathsf{Ra}\mathfrak{D}$ has universe $Nr_2\mathfrak{D}$ and the Boolean operations are the same as those of $\mathcal{D}$, composition ; and converse $\breve{}$ are defined for $a, b \in D$ by [11] 5.3.7:

$$a;b = \mathsf{c}_3(\mathsf{s}_2^1 a \cdot \mathsf{s}_2^0 b) \text{ and } \breve{a} = {}_2\mathsf{s}(0,1)a.$$

Let

$$\eta = [rf(\lambda)]\cdot\pi_{RA}.$$

Recall that λ is an inseparable sentence such that $\lambda \wedge \pi'$ has a model. From the definition of r and f we have $\eta \in \mathsf{RAT}_1$. Let $\mathfrak{Fm}^{\Lambda_3}$ be the algebra of restricted formulas using three variables. Let $\mathcal{F} = \mathfrak{Fr}_1\mathbf{SimRA}$. Let $h : \mathcal{F} \to \mathsf{Ra}\mathfrak{Fm}^{\Lambda_3}$ be the homomorphism that takes the free generator of $\mathcal{F}$ to the formula $'x \in y.'$ Let $\psi = h(\eta)$. Then $\psi \in Fm^{\Lambda_3 \cdot}$ Then we show that ψ is the desired formula. In other words, we show that:

(1) ψ is m consistent,

(2) ψ cannot be extended to a decidable consistent m - complete theory.

We prove (1). It is enough to show that ψ is satisfiable in some model, since then by the soundness of $\vdash_m$ we will have not $\vdash_m \neg\phi$. By our choice of λ and $p_i(x, y)$ $(i \in 2)$ there is a model M such that $\mathsf{M} \models \lambda \wedge \pi'$. But then ψ will be semantically consistent because h preserves meaning. Now we prove (2) Assume seeking a contradiction that T is a recursive complete theory containing ψ. Define

$$R = \{(\tau, \sigma) \in {}^2G : T \vdash_m (h\tau \longleftrightarrow h\sigma)\}.$$

Then R is recursive , since T is such. Also R is a congruence on $\mathcal{F}$, $\eta \in 1/R \neq 0/R$ and $\mathcal{F}/R \in RA$ by $\mathsf{Ra}_{p,m}\mathfrak{Fm}^{\Lambda_3} \in \mathsf{RA}$. (Here $m \geq 4$ plays a crucial role.) Now assume that Λ contains at least one relation symbol R_i of arity ≥ 2. Repeat the proof of the above by writing $R_i(v_0 \ldots v_{\rho_i - 1})$ everywhere in place of $v_0 \in v_1$ and conjuncting the formula

$$R_i(v_0 \ldots v_{\rho_i-1}) \longleftrightarrow \exists v_2 \ldots v_{\rho_i-1} R_i(v_0 \ldots v_{\rho_i-1)}$$

to ψ. And finally we prove that:

Theorem 3.15. *Let* $\mathsf{K} \subseteq \mathsf{RA}$ *be such that* EqK *is recursively enumerable and* $\mathfrak{R}U \in \mathsf{K}$ *for some infinite* U. *Then* $\mathfrak{Fr}\mathsf{K}$ *is not atomic.*

Proof. Let $\lambda \in \mathfrak{Fm}^0_\omega$ be an inseparable formula such that $\lambda \wedge \pi$ has a model. Such a λ exists. Let $\eta = (rf\lambda).\pi_{RA}$. Let $H : \mathcal{F} \to \mathfrak{Fr}_\beta\mathsf{K}$ be a homomorphism such that H maps the free generators of $\mathcal{F}$ to one of the free generators, say g, of $\mathfrak{Fr}_\beta\mathsf{K}$. We show that $H\eta) \neq 0$ and there is no atom in $\mathfrak{Fr}_\beta\mathsf{K}$ below $H\eta$. First we show that $H\eta \neq 0$. By the Löwenheim-Skolem theorems, $\lambda \wedge \pi$ has a model with universe U, say $\mathsf{M} = (U, E) \models \lambda \wedge \pi$. For any $\phi \in Fm^2_\omega$, Let

$$\phi^{\mathsf{M}} = \{(a, b) \in {}^2U : \mathsf{M} \models \phi[a, b]\}.$$

Then $p_i(x, y)^{\mathsf{M}} = kH(rp_i(x, y))$ and $(f\lambda)^M = kH(rf\lambda)$. By $\mathsf{M} \models \pi$ then $kH(\pi_{\mathsf{RA}}) = U \times U$. By $\mathsf{M} \models \lambda \wedge \pi$, we have $(f\lambda)^{\mathsf{M}} = (\lambda)^{\mathsf{M}} = U \times U$. Thus $kH\eta = U \times U$. Assume now that $\mu \leq H\eta$ is an atom in $\mathfrak{Fr}_\beta K$. Define

$T = \{\phi \in Fm^0_\omega : \mu \leq Hrf\phi\}$. Then T is decidable since μ is an atom and EqK is recursively enumerable. Assume that $\lambda \models \phi$. Then $\mathsf{RRA} \models (\pi_{\mathsf{RA}} \cdot \mathsf{rf}\lambda) \leq \mathsf{rf}\phi$, therefore by $\mathsf{QRA} \subseteq \mathsf{RRA}$, it follows that:

$$\mathsf{RA} \models (\pi_{\mathsf{RA}} \cdot \mathsf{rf}\lambda) \leq \mathsf{rf}\phi$$

Hence $\mu \leq rf\phi$ by $\mu \leq \pi_{\mathsf{RA}} \cdot rf\lambda$. Thus $\phi \in T$. Assume that $\lambda \models \neg\phi$. Then $\mu \leq rf(\neg\phi) = -rf\phi$, hence μ is not comparable with $rf\phi$, since $\mu \neq 0$. Thus $\phi \notin T$. The above contradicts the choice of λ, hence there is no atom in $\mathfrak{Fr}_\beta K$ below $H\eta$. □

4 Non atomicity of free algebra of relativized set algebras

The non-atomicity of the finitely generated free algebras of the relativized classes of cylindric algebras remained open for decades. Recently, normal forms were applied in a novel way to attack this problem. It was finally proved in [15] that these free algebras are not atomic. The proof is very involved and contains new techniques that seem to be very powerful. In this section, we will try to survey some of the major ideas of this proof. We will do this by proving the non-atomicity of the diagonal free relativized cylindric algebras. Dropping diagonals simplifies the problem significantly, however it remains hard. The results and the techniques we present here were originally discussed in [27]. We start with giving an axiomatic definition for the class of relativized cylindric algebras. We fix, throughout this section, a finite ordinal $n \geq 2$.

Definition 4.1. A relativized diagonal free algebra of dimension n is an algebraic structure

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_i \rangle_{i \in n}$$

in which $+$ and $\cdot$ are binary operations, and $-$ and c_is are unary operations on A, while 0 and 1 are distinguished elements of A, and which satisfies the following postulates for arbitrary $i \in n$ and $x, y \in A$:

(D0) $\langle A, +, \cdot, -, 0, 1 \rangle$ is Boolean algebra.

(D1) $c_i 0 = 0$.

(D2) $x + c_i x = c_i x$.

(D3) $c_i(x \cdot c_i y) = c_i x \cdot c_i y$.

The class of all relativized diagonal free algebras of dimension n is denoted by Drs_n.

We refer to the above axioms as **Drs**. Let $\mathfrak{S} = \langle S, T_i \rangle_{i<n}$ be any relational structure such that each T_i is a binary relation on S. We define the complex algebra $\mathfrak{Cm}\mathfrak{S} = \langle \mathcal{P}(S), \cup, \cap, \setminus, \emptyset, S, C_i \rangle$, where $\langle \mathcal{P}(S), \cup, \cap, \setminus, \emptyset, S \rangle$ is the Boolean algebra of all subsets of S, and $C_i X = \{y : (x, y) \in T_i \text{ for some } x \in X\}$. We say that $\mathfrak{S}$ is an Drs_n-atom structure if and only if $\mathfrak{Cm}\mathfrak{S}$ is a relativized diagonal free algebra of dimension n. It can be easily proved that $\mathfrak{S}$ is an Drs_n-atom structure if and only if each T_i is an equivalence relation.

In fact, it can be shown that the class Drs_n as a variety is generated by the complex algebras of all Drs_n-atom structures. We will make use of this fact quite frequently. It is obvious that the zero generated fee algebra of Drs_n consists of only two elements, and thus it is atomic. We will prove the following theorem.

Theorem 4.2. *Suppose that $m \geq 1$ is a finite ordinal, then the free algebra $\mathfrak{Fr}_m\mathsf{Drs}_n$ is atomless.*

Fix a finite ordinal $m \geq 1$ and fix a finite non-empty set $X = \{x_0, \ldots, x_{m-1}\}$. We assume that $\mathfrak{Fr}_m\mathsf{Drs}_n$ is freely generated by the variables in X. We suppose that $T(X)$ is the set of all terms that are constructed from X and that there is a standard ordering of all such terms. We use the following abbreviations:

- We write $\tau \approx \sigma$ if $\mathsf{Drs}_n \models \tau = \sigma$, and we write $\tau \not\approx \sigma$ otherwise.
- We also write $\tau \lesssim \sigma$ if and only if $\tau + \sigma \approx \sigma$.
- Let $\mathfrak{S} = \langle S, T_i \rangle_{i \leq n}$ be an Drs_n-atom structure, and let $\iota : X \to \mathcal{P}(S)$ be an evaluation. For each $v \in S$, we write $(\mathfrak{S}, v, \iota) \models \tau$ iff v is an element of the interpretation of τ in $\mathfrak{Cm}\mathfrak{S}$ under the evaluation ι.
- Let $\sum$ and $\prod$ be the generalized versions of $+$ and $\cdot$, respectively.

Let $k \in \omega$, we define the following inductively:

- N_0 is the set of all terms $X^\alpha \overset{\text{def}}{=} x_0^\alpha \cdot x_1^\alpha \cdots x_m^\alpha$ where each x_i^α, $i \leq m$, is either x_i or $-x_i$.
- Suppose that $\sigma_0, \sigma_1, \ldots, \sigma_h$ are the members of N_k in the standard order. Then N_{k+1} is the set of all terms of the form

$$X^\alpha \cdot \prod_{i=0}^{n-1} (c_i\sigma_0)^\beta \cdot (c_i\sigma_1)^\beta \cdots (c_i\sigma_h)^\beta$$

  where $X^\alpha \in N_0$ and, for each $i < n$ and each $j \leq h$, $(c_i\sigma_j)^\beta$ is either $c_i\sigma_j$ or $-c_i\sigma_j$.

The normal forms of degree 0 describe the state of the free generators. The normal forms of degree $k+1$ describe the state of the free generators and the terms of the forms $c_i\sigma$ with $i < n$ and σ a normal form of degree k. For more details, we refer the reader to [17].

Lemma 4.3. *Let $\tau \in T(X)$ be such that $\tau \not\approx 0$. Then there is a finite ordinal $q \in \omega$ and a non-empty finite set $S \subseteq N_q$ of normal forms of degree q such that $\tau \approx \sum S$.*

Proof. The proof can be established by a simple induction argument on the complexity of the term τ. We omit the details of this proof because it is routine. For the full proof, one can see [17, Theorem 4.10]. □

We need to introduce some notations that will be used in the proofs. Let $k \in \omega$, let $\tau \in N_{k+1}$ and let $i < n$. We define $sub_i(\tau)$ to be the set of all forms $\sigma \in N_k$ for which $c_i\sigma$ appears without negation in the conjuncts of τ. We also define $color_n(\tau)$ to be the set of all free generators that appears without negation in the conjuncts of τ. The following observations can be easily checked:

Let $k, l \in \omega$ be finite ordinals such that $k \leq l$, and let $i < n$ be arbitrary.

1. Let τ and σ be two different normal forms of degree k, then $\tau \cdot \sigma \approx 0$.
2. Suppose that $\sigma_0, \sigma_1, \ldots, \sigma_h$ are the members of N_k in the standard order. Then,
$$\sigma_0 + \sigma_1 + \cdots + \sigma_h \approx 1.$$
3. For each $\tau \in N_l$, if $\tau \not\approx 0$, then there is a unique normal form $\sigma \in N_k$ such that $\tau \lesssim \sigma$.
4. Let $\tau \in N_{k+1}$ and let $\sigma \in N_k$ be such that $\tau \not\approx 0$ and $\tau \lesssim \sigma$. Then, we have $\sigma \in sub_i(\tau)$.
5. Let $\sigma \in N_k$, $\tau \in N_{k+1}$ and $\tau' \in N_{l+1}$ be normal forms of the indicated degrees. If $\tau' \not\approx 0$, $\tau' \lesssim \tau$ and $\sigma \in sub_i(\tau)$, then there exists $\sigma' \in N_l$ such that $\sigma' \lesssim \sigma$.
6. Let $\sigma \in N_l$, $\tau \in N_{l+1}$ and $\tau' \in N_{k+1}$ be normal forms of the indicated degrees. If $\tau \not\approx 0$, $\tau \lesssim \tau'$ and $\sigma \in sub_i(\tau)$, then there is $\sigma' \in N_k$ such that $\sigma \lesssim \sigma'$.

We refer to the above observations as **NF**. The first two observations represent the fact that the set of all normal forms of the same degree form a partition of the unit of the free algebra $\mathfrak{Fr}_m\mathsf{Drs}_n$.

To proof that the free algebra $\mathfrak{Fr}_m\mathsf{Drs}_n$ is not atomic, it is enough to prove that no normal form can be an atom in $\mathfrak{Fr}_m\mathsf{Drs}_n$. Fix a finite number $q \geq 1$ and a normal form $\tau \in N_q$. Assume that $\tau \not\approx 0$. We will construct a finite atom structure $\mathfrak{S}$ that witnesses the satisfiability of the form τ. Then we will extend $\mathfrak{S}$ in two different ways showing that τ cannot be an atom in the free algebra $\mathfrak{Fr}_m\mathsf{Drs}_n$.

The construction of $\mathfrak{S}$ goes inductively through $q + 1$-many steps. In each step, we construct an atom structure that extends all the atom structures constructed in the previous steps. Whenever we add a node (an element) to any of these atom structures, we label it with a normal form.

Step 0:

Pick up a node u, and assign to it the label $tag(u) = \tau$. Now, for each $i < n$, define the sets $S_0 = S_0^i = \{u\}$ and $T_i^0 = \{(u, u)\}$. We also define $\mathfrak{S}_0 \stackrel{\text{def}}{=} \langle S_0, T_i^0\rangle_{i<n}$.

Step 1:

Let U be an infinite set such that $u \notin U$. For each $i < n$, construct an injective function,

$$\psi_u^i : sub_i(\tau) \to U,$$

such that $Rng(\psi_u^i)$s are finite and pairwise disjoint. Let $i < n$ be arbitrary.

- Let $S_1^i = Rng(\psi_u^i)$ and $S_1 = S_1^0 \cup \cdots \cup S_1^{n-1}$.
- For any $v \in S_1^i$, let $tag(v) = (\psi_u^i)^{-1}(v)$.
- Let $T_i^1 = \{(v, w) : v, w \in S_1^i \cup \{u\}\}$.
- We define $\mathfrak{S}_1 \stackrel{\text{def}}{=} \langle S_1, T_i^1\rangle_{i<n}$.

From step t to step t+1:

More generally, suppose that $q \geq 2$, and $S_t, S_t^0, \ldots, S_t^{n-1}$ have been constructed and the labelling tag has been extended to cover S_t, for some $1 \leq t \leq q-1$. Let $i < n$ be arbitrary. For every every $j < n$ with $j \neq i$ and every $v \in S_t^j$, create an injective function

$$\psi_v^i : sub_i(tag(v)) \to U \setminus S_t,$$

such that the ranges of all those functions ψ_v^is are finite and pairwise disjoint. Let

- Let $S_{t+1}^i = \bigcup\{Rng(\psi_v^i) : v \in S_t^j, \text{ for some } j < n \text{ with } j \neq i\}$, and $S_{t+1} = S_t \cup S_{t+1}^0 \cup \cdots \cup S_{t+1}^{n-1}$.

- For any $j < n$ with $j \neq i$, any $v \in S_t^j$ and any $w \in Rng(\psi_v^i)$, let $tag(w) = (\psi_v^i)^{-1}(w)$.
- Let $T_i^{t+1} = T_i^t \cup \{(w_1, w_2) : w_1, w_2 \in (Rng(\psi_v^i) \cup \{v\}), v \in S_t^j$ and $j < n$ with $j \neq i\}$.
- Define $\mathfrak{S}_{t+1} \stackrel{\text{def}}{=} \langle S_{t+1}, T_i^{t+1} \rangle_{i<n}$.

Note that, throughout the construction of $\mathfrak{S}_{t+1}$, there are no i-connections made between the nodes in S_{t-1} and the nodes in S_{t+1}, moreover, the new i-connections are only added between the nodes of the form $v \in S_t^j$, for some $j < n$ with $j \neq i$, and the nodes $w \in Rng(\psi_v^i)$. Moreover, all the normal forms labelling the nodes in $S_{t+1} \setminus S_t$ are of the same degree $q - (t + 1)$, which is one less than the degree of the normal forms labelling the nodes in $S_t \setminus S_{t-1}$. We continue in the same manner till we get the following sequence of atom structures

$$\mathfrak{S}_0 \subseteq \cdots \subseteq \mathfrak{S}_q.$$

Let $S \stackrel{\text{def}}{=} S_q$. For each $i < n$, it is clear that the set $T_i \stackrel{\text{def}}{=} T_i^q$ is an equivalence relation on the set S. Thus, the structure $\mathfrak{S} \stackrel{\text{def}}{=} \langle S, T_i \rangle_{i<n}$ is an Drs_n-atom structure. Define the evaluation $\iota : X \to \mathcal{P}(S)$ as follows: For each $x \in X$, let $\iota(x) = \{v \in S : x \in color(tag(v))\}$.

Lemma 4.4. *For each $v \in S$, we have $(\mathfrak{S}, v, \iota) \models tag(v)$.*

Proof. Let $v \in S$ be arbitrary. We note that $tag(v) \not\approx 0$ because of **Drs** and the assumption that $\tau \not\approx 0$. Let $t \leq q$ and suppose that $v \in S_t$ and $v \notin S_{t-1}$ if $t \geq 1$. By construction, we know that $tag(v) \in N_{q-t}$. For every $h \leq q$, we define $tag_h(v)$ as follows: If $h \geq q - t$, then we define $tag_h(v) = tag(v)$. Otherwise, let $tag_h(v)$ be the unique normal form in N_h for which $tag(v) \lesssim tag_h(v)$. It is enough to prove the following:

$$(\forall h \leq q)\ (\forall v \in S) \quad (\mathfrak{S}, v, \iota) \models tag_h(v). \tag{1}$$

We prove (1) by induction on h. By the construction of the atom structure $\mathfrak{S}$ and the choice of the evaluation ι, the following is true: For any $v \in S$ and any $x \in X$,

$$(\mathfrak{S}, v, \iota) \models x \iff x \in color(tag(v)). \tag{2}$$

Thus, it is clear that the instance of (1) when $h = 0$ holds. Let $h \leq q - 1$ and assume that for every $v \in S$, $(\mathfrak{S}, v, \iota) \models tag_h(v)$. Let $v \in S$ be arbitrary but fixed, we need to show that $(\mathfrak{S}, v, \iota) \models tag_{h+1}(v)$. Let $t \leq q$ and $i < n$ be such that $v \in S^i_t$. If $h \geq q - t$, then we are done by the induction hypothesis. So, suppose that $h < q - t$, then $t \leq q - 1$ and $tag_{h+1}(v) \in N_{h+1}$. Let $j < n$ and $\sigma \in N_h$, we need to prove that,

$$\sigma \in sub_j(tag_{h+1}(v)) \iff (\mathfrak{S}, v, \iota) \models c_j\sigma. \tag{3}$$

We start with the direction $\Longrightarrow$. Suppose that $\sigma \in sub_j(tag_{h+1}(v))$. Recall that $tag(v) \lesssim tag_{h+1}(v)$, then there exists $\sigma' \in sub_j(tag(v))$ such that $\sigma' \lesssim \sigma$. We have one of the following cases:

- Assume that $j \neq i$, then by the construction of $\mathfrak{S}$, there exists $w \in Rng(\psi^j_v) \subseteq S^j_{t+1}$ such that $(v, w) \in T_j$ and $tag(w) = \sigma'$. By the induction hypothesis, we have $(\mathfrak{S}, w, \iota) \models tag_h(w) = \sigma$, and consequently, $(\mathfrak{S}, v, \iota) \models c_j\sigma$.
- Assume that $j = i$. If $t = 0$, then by the same argument one can show that $(\mathfrak{S}, v, \iota) \models c_i\sigma$. So, suppose that $t \geq 1$. Let $w \in S_{t-1}$ be such that $v \in Rng(\psi^i_w)$, then $(w, v) \in T_i$. By **Drs**, we have

  $$0 \not\approx tag(w) = tag(w) \cdot c_i tag(v) \lesssim tag(w) \cdot c_i c_i \sigma' \lesssim tag(w) \cdot c_i\sigma'.$$

  Thus, by Lemma 4.3 and **Drs**, there is $\sigma'' \in sub_i(tag(w))$ such that $\sigma'' \lesssim \sigma'$. Thus, by the above item, there is a node $v' \in Rng(\psi^i_w) \cup \{w\}$ such that $\{(w, v'), (v, v')\} \subseteq T_i$ and $tag_{q-t}(v') = \sigma''$. So by the induction hypothesis, we have $(S, v, \iota) \models ta_h(v') = \sigma$. Thus, $(\mathfrak{S}, v, \iota) \models c_i\sigma$ as desired.

Now, we consider the other direction $\Longleftarrow$. Suppose $\sigma \notin sub_j(tag_{h+1}(v))$ and assume towards a contradiction that there is $w \in S$ such that $(v, w) \in T_j$ and $(\mathfrak{S}, w, \iota) \models \sigma$.

- Suppose that $w = v$, then by the induction hypothesis we have $\sigma = tag_h(v)$, which makes a contradiction with the assumption that $\sigma \notin sub_j(tag_{h+1}(v))$ and **NF**.
- Suppose that $j \neq i$ and $w \neq v$. Recall that $v \in S^i_t$, hence by the construction, there is $\sigma' \in sub_j(tag(v))$ such that $tag(w) = \sigma'$. By the induction hypothesis, $\sigma' \approx tag(w) \lesssim tag_h(w) = \sigma$. Then, by **NF**, we must have $\sigma \in sub_j(tag_{h+1}(v))$. This contradicts the assumptions.
- Suppose that $j = i$ and $w \neq v$. If $t = 0$, then, similarly to the above item, we can easily reach a contradiction. Suppose that $t \geq 1$, and let $v' \in S_{t-1}$ be the unique node such that $v \in Rng(\psi^i_{v'})$ and either $w = v'$ or $w \in Rng(\psi^i_{v'}) \subseteq S^i_t$. Hence, $tag(w)$ is a normal form of degree either

$q-t+1$ or $q-t$. By the construction of $\mathfrak{S}$ and **NF**, there is a normal form $\sigma' \in sub_i(tag(v'))$ such that $tag(w) \lesssim \sigma'$. But $(\mathfrak{S}, w, \iota) \models \sigma$ implies that $tag(w) \lesssim \sigma' \lesssim tag_h(w) \lesssim \sigma$. So, we reach a contradiction again.

Thus, we have shown that (3) is true. Now, (2) and (3) imply that $(\mathfrak{S}, v, \iota) \models tag_{h+1}(v)$. Therefore, by the induction principle, (1) is established as desired. □

Now, we are ready to extend the atom structure $\mathfrak{S}$. We do this as follows. Let $v_0 = u \in S_0$ be the unique node in S_0 whose label is τ. By **NF**, there is a unique $\tau_1 \in N_{q-1}$ such that $\tau \lesssim \tau_1$ and $\tau_1 \in sub_0(\tau)$. By the construction of $\mathfrak{S}$, there is $v_1 \in Rng(\psi^0_{v_0})$ such that $tag(v_1) = \tau_1$ and $(v_0, v_1) \in T_0$. If $q \geq 2$, then again there is a unique $\tau_2 \in N_{q-2}$ such that $\tau_1 \lesssim \tau_2$ and $\tau_2 \in sub_1(\tau_1)$. Then there is $v_2 \in Rng(\psi^1_{v_1})$ such that $tag(v_2) = \tau_2$ and $(v_1, v_2) \in T_1$. Continuing this manner, we get a sequence $v_0, \ldots, v_q$ satisfying the following:

(a) For every $t \leq q$, $tag(v_t) \in N_{q-t}$.

(b) For every $t < q$, if t is even then $(v_t, v_{t+1}) \in T_0$ and, if t is odd then $(v_t, v_{t+1}) \in T_1$.

Without loss of generality we may assume that q is even. Let $z \notin S$ be a brand new element. Define the extended structure $\mathfrak{S}^+ = \langle S^+, T_i^+ \rangle_{i<n}$, where $S^+ = S \cup \{z\}$ and, for any $1 < i < n$, $T_i^+ = T_i$ and

$$T_0^+ = T_0 \cup \{(z, z), (z, v_q), (v_q, z)\}.$$

We extend the *tag*s to z as follows. Choose a normal form $\varsigma \in N_0$ which is different than $tag(v_q) \in N_0$. This is possible because of the assumption that $m > 1$. Now, let $tag(z) = \varsigma$. Define the evaluation $\iota^+ : X \to \mathcal{P}(S^+)$ as follows: For each $x \in X$, let $\iota^+(x) = \{v \in S^+ : x \in color(tag(v))\}$. Similarly to Lemma 4.4, one can prove

$$\text{For any } v \in S \quad (\mathfrak{S}^+, v, \iota^+) \models tag(v). \tag{4}$$

Moreover, since in $\mathfrak{S}$ the node v_q is 0-connected only to itself, we have

$$(\mathfrak{S}, v_q, \iota) \models tag(v_q) \cdot -c_0\varsigma \quad \text{and} \quad (\mathfrak{S}^+, v_q, \iota^+) \models tag(v_q) \cdot c_0\varsigma \tag{5}$$

Proof of Theorem 4.2. We prove that τ is not an atom. Remember the sequence $v_0, \ldots, v_q$. For every $t \leq q$, there are unique normal forms $\theta_t, \pi_t \in N_{q-t+1}$ such that $(\mathfrak{S}, v_t, \iota) \models \theta_t$ and $(\mathfrak{S}^+, v_t, \iota^+) \models \pi_t$. Note that $tag(v_t) \in N_{q-t}$, thus

$$0 \not\approx \theta_t \leq tag(v_t) \quad \text{and} \quad 0 \not\approx \pi_t \leq tag(v_t). \tag{6}$$

It remains to prove the following:

$$\text{For every } t \leq q, \quad \theta_t \cdot \pi_t \approx 0. \tag{7}$$

We use induction on $q-t$. By (5), it is clear that $\theta_q \cdot \pi_q \approx 0$. Assume that $\theta_{q-t} \cdot \pi_{q-t} \approx 0$ for some $t < q$. Let $i < 2$ be such that $i = q-t-1$ $(mod\ 2)$. Remember $(v_{q-t-1}, v_{q-t}) \in T_i$ and $\theta_{q-t}, \pi_{q-t}, tag(v_{q-t-1}) \in N_{t+1}$. But $\theta_{q-t} \cdot \pi_{q-t} \approx 0$, by the induction hypothesis. Hence, and without loss of generality, we may assume that $\theta_{q-t} \cdot tag(v_{q-t-1}) \approx 0$. Now, since $(v_{q-t-1, v_{q-t}}) \in T_i$, we have

$$(\mathfrak{S}, v_{q-t-1,\iota}) \models \theta_{q-t-1} \cdot c_i\theta_{q-t}. \tag{8}$$

We need to show that $(\mathfrak{S}^+, v_{q-t-1,\iota^+}) \models -c_i\theta_{q-t}$. For this, let $v \in S^+$ be such that $(v_{q-t-1,v}) \in T_i^+$. Then, by constructions, $v \in S$, $(v_{q-t-1,v}) \in T_i$ and v is either v_{q-t-1} itself or $v \in Rng(\psi^i_{v_{q-t-1}})$.

- Suppose that $v = v_{q-t-1}$. Recall the assumption that $\theta_{q-t} \cdot tag(v_{q-t-1}) \approx 0$. Thus, $(\mathfrak{S}, v, \iota) \not\models \theta_{q-t}$.
- Suppose that $v \in Rng(\psi^i_{v_{q-t-1}})$ and $v \neq v_{q-t}$. By the injectivity of $\psi^i_{v_{q-t-1}}$ and **NF**, we have $tag(v) \cdot tag(v_{q-t}) \approx 0$. But, by (6), $\theta_{q-t} \lesssim tag(v_{q-t})$. Hence, by (4), we have $(\mathfrak{S}, v, \iota) \not\models \theta_{q-t}$.
- Suppose that $v = v_{q-t}$. Then, by the induction hypothesis, we have $(\mathfrak{S}, v, \iota) \not\models \theta_{q-t}$.

Whence,

$$(\mathfrak{S}^+, v_{q-t-1}, \iota^+) \models \pi_{q-t-1} \cdot -c_i\theta_{q-t}. \tag{9}$$

Thus, $\theta_{q-t-1} \lesssim c_i\theta_{q-t}$ but $\pi_{q-t-1} \lesssim -c_i\theta_{q-t}$. In other words, $\theta_{q-t-1} \cdot \pi_{q-t-1} \approx 0$. So, (7) follows immediately by the induction principle. Therefore, by (6) and (7), each $tag(v_t)$, $t \leq q$, is not an atom in the free algebra $\mathfrak{Fr}_m\mathsf{Drs}_n$. The desired follows from the fact that $tag(v_0) = \tau$. □

Mohamed Khaled [15] proved that the free Crs_ns with finitey many generators are not atomic and developed his proof to show that many guarded logics has a form of Gödel's incompleteness property: There is a formula ϕ (in the signature of the guarded logic under investigation) that cannot be extended to a finite complete recursive theory. Such 'incompletable formulas are called *inseparable formulas* by Németi as pointed out above. This solves a long standing open problem in algebraic logic posed by Németi in the early eighties of the last century. Similar results can be found in [20, 21, 16, 27].

This, in turn, shows that after all 'the taken for granted' implication 'Gödel's incompleteness $\implies$ undecidability' does not always hold. Results of this kind are open to huge philosophical considerations, revisions and repercussions, and are far from being fully understood. Indeed such results tend to raise more questions than answers. The question that bears a lot of discussion and reflection in this connection is how faithfull the algebraic translation to non-atomicity of the free algebras, vis a vis the in built dependence of the proof of Gödel's incompleteness Theorem in Peano arithmetic (without the axiom of infinity) using Gödel numbering. Is Gödel's celebrated incompleteness theorem intrinsic to arithmetic (and richer formal systems like set theory), or can it lend itself to different, possibly more general frameworks? Is the idea of Gödel numbering–mirroring statements about numbers to statements about other statements of numbers, possibly themselves- applicable only to the entities known as natural numbers? The two ingenious components in Gödel's proof are *diagonalization* and *self-reference*. Such methods and antimonies were known before Gödel. Diagonalization is implemented in Cantor's proof of the uncountability of $\mathbb{R}$ and self reference appeared (philosophically) with the liar paradox, later getting a more mathematical manifestation in the famous hugely influential Russell's parardox with several scattered re-incarnations in interdisciplinary literature between mathematics, logic and philosophy [33]. But combining the two is certainly a master stroke proving one of the most important Theorems in mathematics in the 20th century and beyond. These two ingredients of Gödel's proof simply vanish in the algebraic proofs for Crs_n proving the non-atomicity of their finitely generated free algebras. The new proofs use an ingenious purely algebraic method [15, 16, 27]. The consistency of a form of Gödel's incompleteness theorem and decidability for guarded fragments of L_n is certainly an exciting and a telling co-existence.

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Ok, GÃ¶delâ€™s Incompleteness Theorem is not only one of the most important pieces of mathematics ever produced, but also one thatâ€™s profoundly hard to understand. For that reason, Iâ€™m not going to explain how exactly it works, but rather give an outline of the idea behind it. Letâ€™s start off with the basics: Whatâ€™s a mathematical sentence? In mathematics every statement has only one of two answers: True and False. 9 is a prime number. In 1931 GÃ¶del published his first incompleteness theorem, â€œÃœber formal unentscheidbare SÃ¤tze der Principia Mathematica und verwandter Systemeâ€ (â€œOn Formally Undecidable Propositions of Principia Mathematica and Related Systemsâ€), which stands as a major turning point of 20th-century logic. This theorem established that it is impossible to use the axiomatic methodâ€¦ Read More. formalism. In positivism: Developments in linguistic analysis and their offshoots. Before GÃ¶delâ€™s discovery, it had seemed plausible that a mathematical system could be complete in the sense that any well-formed formula of GÃ¶del's incompleteness theorems is the name given to two theorems (true mathematical statements), proved by Kurt GÃ¶del in 1931. They are theorems in mathematical logic. Mathematicians once thought that everything that is true has a mathematical proof. A system that has this property is called complete; one that does not is called incomplete. Also, mathematical ideas should not have contradictions. This means that they should not be true and false at the same time. A system that does not include 2 Generalized GÃ¶del's second incompleteness theorem. 3 Optimality of the GÃ¶del's second incompleteness theorem. 4 Concluding Remarks.Â Abstract GoÂ¨delâ€™s second incompleteness theorem is generalized by showing that if the set of axioms of a theory T âŠ‡ PA is Î£n+1-denable and T is Î£n-sound, then T dose not prove the sentence Î£n-Sound(T) that expresses the Î£n-soundness of T . The optimal-ity of the generalization is shown by presenting a Î£n+1-denable. (indeed a complete âˆ†n+1-denable) and Î£nâˆ'1-sound theory T such that PA âŠ† T and Î£nâˆ'1-Sound(T) is provable in T . It is also proved that no recursively enumerable and Î£1-sound theory of arithmetic, even very weak theories which do not contain Robinsonâ€™s Arithmetic, can prove its own