

The concept of axiom in Hilbert's thought

Giorgio Venturi

1 Introduction

David Hilbert is considered the champion of formalism and the mathematician who turned the axiomatic method into what we know nowadays. Hilbert compares in 1900¹ the axiomatic method with the genetic one, and he argues that only the former is capable of giving a true foundation to mathematics.

In this paper I will analyze the key concept of the axiomatic method, namely that of axiom.

To understand the concept of axiom in Hilbert's work means to delve into the modern axiomatic method. We will also explore the goals of Hilbert's use of the axiomatic method and its differences from previous practice.

An important difference between the axiomatic method at the beginning of the 20th century and the ancient one is the role of intuition. 20th century mathematics needed to account for the abstract features that were introduced during the 19th century. On the other hand, previous mathematics was rather based on intuitive skills that were not considered sufficient anymore.

In Hilbert's foundational writings we can find a notion of formal system that is already similar to the one we use today, and in the *Grundlagen der Geometrie*² (1899) the new features are implicit yet. Later on, formal systems gain some structural characters that depend on the use of a new formal logic. Hilbert spells out the tools of logic inferences, proofs are regarded as finite objects, theorems are now considered valid only if they are deducible from the axioms with a finite number of admitted inferences, etc. As a consequence we can find a deep change in the concept of axiom.

2 Two different concepts of axiom

Hilbert's philosophical papers on the foundation of mathematics can be divided into two periods though not neatly separated. Their content differs in what is meant to be the nature of the axioms and in the methods used to achieve certainty in mathematics.

¹In [Hilbert 1900a].

²[Hilbert 1899].

The first period centers around his work in geometry and his sketched attempt to prove the consistency of a weak form of arithmetic³. In the works of this period, Hilbert's concept of axiom is linked to a "deepening of the foundations of the individual domains of knowledge"⁴. Indeed the axiomatization of a theory is gained by making explicit the logical structure of the corresponding domain of knowledge.

When we are engaged in investigating the foundations of a science, we must set up a system of axioms which contains an exact and complete description of the relations subsisting between the elementary ideas of that science⁵.

Moreover axioms are viewed as the logical means used to fix the basic principles of a science and so, in Hilbert's words,

The procedure of the axiomatic method, as it is expressed here, amounts to a *deepening of the foundations* of the individual domains of knowledge⁶.

The second period starts after Hilbert has resumed the study of the foundation of mathematics. In this period's works we can see an effort to build the whole of mathematics on few axioms. These axioms were supposed to gain their legitimacy from the new *proof theory*, that, following Hilbert, "make[s] a protocol of the rules according to which our thinking actually proceeds"⁷. As a matter of fact, these rules are the a priori component of any form of mathematical knowledge. Indeed

also [...] mathematical knowledge in the end rests on a kind of intuitive insight [*anschaulicher Einsicht*] of this sort, and even that we need a certain intuitive *a priori* outlook for the construction of number theory⁸.

As a consequence, considering the whole of the mathematics as a formal system, the choice of its axioms rests on the individuation of the *a priori* principles that governs our conceptual knowledge and our mathematical experience.

³See [Hilbert 1905].

⁴[Hilbert 1918].

⁵[Hilbert 1900].

⁶[Hilbert 1918].

⁷[Hilbert 1930].

⁸[Hilbert 1930].

2.1 The first period

In the first period Hilbert succeeds in completing the axiomatization of geometry and of the theory of real numbers. As a consequence the example of geometry is fundamental in Hilbert's reflections. In this period the enquiry about axioms is seen as a way to delve into the logical relationship among theorems of a theory. Here "theory" is not used in the formal sense, but it refers to any mathematical field of research that features only one subject of enquiry and homogeneous methods. In Hilbert's words:

On the contrary I think that wherever, from the side of the theory of knowledge or in geometry, or from the theories of natural or physical science, mathematical ideas come up, the problem arises for mathematical science to investigate the principles underlying these ideas and so to establish them upon a simple and complete system of axioms, that the exactness of the new ideas and their applicability to deduction shall be in no respect inferior to those of the old arithmetical concepts⁹.

The analysis of the basic principles of a theory on the one hand leads to the choice of the axioms, and on the other hand it defines the concepts and relations in play.

When we are engaged in investigating the foundations of a science, we must set up a system of axioms which contains an exact and complete description of the relations subsisting between the elementary ideas of that science. The axioms so set up are at the same time the definitions of those elementary ideas; and no statement within the realm of the science whose foundation we are testing is held to be correct unless it can be derived from those axioms by means of a finite number of logical steps¹⁰.

We now see that in this first period Hilbert has a precise idea of what axioms are: they are *implicit definitions*. Axioms define basic concepts and relations of a theory. Moreover the process of formalization is complete only when the definitions are properly given, through the axioms, and no other characteristic note can be added.

At this point, Hilbert has not handled the problem of the formalization of logic yet, nor Russell and Whitehead have written the *Principia mathematica*. For this reason it is clear that this "completeness" Hilbert talks about has nothing to do with the completeness of logic or of the deductive methods. Rather in the search for completeness we can see a longing for

⁹[Hilbert 1900].

¹⁰[Hilbert 1900].

some sort of categoricity. The unique sought for model is a realization of what intuition dictates.

In 1899-1900 Hilbert explicitly mentions the role of both intuition and axioms.

These axioms may be arranged in five groups. Each of these groups expresses, by itself, certain related fundamental facts of our intuition¹¹.

Also:

The use of geometrical signs as a means of strict proof presupposes the exact knowledge and complete mastery of the axioms which underlie those figures; and in order that these geometrical figures may be incorporated in the general treasure of mathematical signs, there is necessary a rigorous axiomatic investigation of their conceptual content. [...] so the use of geometrical signs is determined by the axioms of geometrical concepts and their combinations¹².

From the passage quoted above it seems that the axiomatic method proceeds by analyzing theorems and concepts that constitute a mathematical theory. The purpose is to isolate the basic principles that correspond to the intuitive ideas of the mathematical entities involved. Then these principles are formalized in the form of axioms that can *completely* define the concepts of a theory, if the process has been properly followed.

Hence the axiomatic method, by means of formalization, enters into play after the informal development of a theory. This idea is still present in *Axiomatisches Denken*, in 1917, and at the beginning of the Twenties, when Hilbert still believed that the formalization of logic was over, thanks to *Pincipia mathematica*

The actual so-called *axioms* of geometry, arithmetic, statics, mechanics, radiation theory, or thermodynamics arose in this way. These axioms form a layer of axioms which lies deeper than the axiom-layer given by the recently-mentioned fundamental theorems of the individual field of knowledge. The procedure of the axiomatic method, as it is expressed here, amounts to a *deepening of the foundations* of the individual domains of knowledge – a deepening that is necessary to every edifice that one wishes to expand and to build higher while preserving its stability¹³.

¹¹[Hilbert 1899].

¹²[Hilbert 1900]. Here the notion of sign is wider than in the second period and geometrical figures are considered as signs.

¹³[Hilbert 1918].

The remarks presented so far undermine the claim that Hilbert is a champion of formalism, at least in two respects. First, the axiomatic method does not require a formalized logic, not even symbols and proceeds by trial and error. And, if used, formalization individuates and represents the basic concepts of a mathematical theory by means of symbols that have an intuitive content grasped by mathematicians.

Before the formalization Hilbert sees a *historical* development on whose basis the mathematician recognizes the intuitive content of the signs. This ensures that the formalization has got a *meaning*.

In the next section I will discuss what Hilbert means by “intuitive content of the symbols”. I will also examine how the intuitive content of the symbols links the demonstrative use of a formal system to the meaning of the concepts of a theory.

The analysis of these issues will provide a better understanding of the reason why a consistency proof is, according to Hilbert, the only possible way to establish the truth of the axioms. Moreover Hilbert also argues that axioms, insofar implicit definitions, also guarantee the existence of the entities corresponding to the concepts defined by them, provided the axioms are mutually consistent.

In we want to investigate a given system of axioms according to the principles above, we must distribute the combinations of the objects taken as primitive into two classes, that of entities and that of nonentities, with the axioms playing the role of prescriptions that the partition must satisfy¹⁴.

In *Mathematischen Probleme* Hilbert states his famous equation of consistency with existence:

In the case before us, where we are concerned with the axioms of real numbers in arithmetic, the proof of the compatibility of the axioms is at the same time the proof of the mathematical existence of the complete system of real numbers or of the continuum¹⁵.

Since consistency and existence are so strongly linked, we could ask which role the axioms play in this relationship. We saw that, for Hilbert, existence depends on axioms. Moreover in case the axiomatization of a theory is complete, the entities defined by the axioms are uniquely determined and hence their existence would be stable, that is: since no new axioms can be added, because of the completeness, as a consequence the existence of the entities cannot change. Indeed, in a letter to Frege, with regard to the

¹⁴[Hilbert 1905].

¹⁵[Hilbert 1900].

possibility of extending the axioms of a given theory, Hilbert says: “every axiom takes part in the definition of the concept, and so every new axiom changes the concept”¹⁶. Then Hilbert holds a substantial coincidence between existence and existence within a formal system, as we can read in the following passage:

[...] we can not have the complete definition of the concept of point, before the construction of the system of axioms is over¹⁷.

On the other hand the problem that there might be different possible coherent formalizations of the same theory does not concern Hilbert. As a matter of fact, Hilbert acknowledges that the question whether different axiomatic systems can both be legitimate is theoretically interesting. However Hilbert does not explain how it is possible that two different theories can talk about the same things, since different axioms define different concepts. It is possible, but we are now entering the realm of speculation, that Hilbert would have replied that there are different way to define the same concept, although the syntactical presentations can differ. In these observations we can see a weak form a realism in Hilbert’s ideas, since the existence of abstract entities does not depends on the symbols we use to express them. Nevertheless, in Hilbert’s works we cannot find an answer to this problem, but it is reasonable to think that Hilbert believed in the existence of a complete formalization of a given concept, capable of uniquely determining that concept and making it existent¹⁸. This idea is coherent with his confidence that every mathematical problem has a unique solution.

On this respect there is another question that is left open by Hilbert: what kind of entities are defined by the implicit definitions of the axioms? This problem was explicitly raised by Frege in the a latter to Hilbert:

The characteristic marks you give in your axioms are apparently all higher than first-level; i.e., they do not answer to the question “What properties must an object have in order to be a point (a line, a plane, etc.)?”, but they contain, e.g., second-order relations, e.g., between the concept *point* and the concept *line*. It seems to me that you really want to define second-level concepts but do not clearly distinguish them first-level ones¹⁹.

Indeed Hilbert is not precise in saying what the axioms define, sometimes they seems to define mathematical object:

¹⁶Letter from Hilbert to Frege December 29th, 1899; in [Frege 1980].

¹⁷Letter from Hilbert to Frege December 29th, 1899; in [Frege 1980].

¹⁸At the end we will see that the formalization of a concept can be incomplete. So, even if the existence of a concept is stable, the formal system and the implicit definitions given by the axioms are dynamic and potentially an open processes.

¹⁹Letter from Frege to Hilbert January 6th, 1900; in [Frege 1980].

I regard my explanation in sec. 1 as the definition of the concepts point, line, plane - if one adds again all the axioms of groups I to V as characteristic marks.²⁰

Sometimes he says that axioms define the relations between mathematical objects. As a matter of fact, in the same letter to Frege, Hilbert says that the axioms of the *Grundlagen der Geometrie* can also define the concept of “between”. In defense to Hilbert it is worth saying that at that time there was no clear distinction between first order and, if not second order, stronger logics. In the following we will see that Hilbert will abandon this careless ontological commitments and will consider the axioms just as defining the relations between mathematical concepts, or as “images of thoughts”. Even if Hilbert chose not to reply to the objections of Frege, we could argue that he indeed carefully read them.

In this first period Hilbert thinks also that it is also necessary to prove the independence of the axioms. The proof, once obtained, would be a point in favor of the adequacy of the choice of the axioms, rather than in favor of their truth. Indeed if the search for axioms is an analysis of the basic principles of a theory, an independence proof would mean that the analysis has been accurate and has been able to single out the right basic principles of the theory.

Upon closer consideration the question arises *whether, in any way, certain statements of single axioms depend upon one another, and whether the axioms may not therefore contain certain parts in common, which must be isolated if one wishes to arrive at a system of axioms that shall be altogether independent of one another*²¹.[.]

This need of an independence proof, unlike that of a consistency proof, will be dropped in the second period. This is another hint of a change in the concept of axiom.

2.2 The second period

Hilbert’s second period begins, publicly, in the early Twenties, but its roots can be traced back to the last years of the previous decade. Different reasons concur in indicating a change of opinions and attitudes towards the foundation of mathematics. Hilbert abandons the confidence in the systematization of logic proposed by Russell and Whitehead and expounded in the *Principia mathematica*. Hilbert then starts the most original contribution to the study of logic, in order to improve its formalization. Moreover in that

²⁰Letter from Hilbert to Frege December 29th, 1899; in [Frege 1980].

²¹[Hilbert 1900]

period the debate around intuitionism becomes more and more controversial. This leads Hilbert to often intervene in the debate against the irrational pushes that animated the mathematical community in those years.

In the lectures *Neubergründung der Mathematik. Erste Mitteilung* (1922) and *Die logischen Grundlagen der Mathematik* (1923) Hilbert outlines a new analysis of the concept of axiom. These two works can be regarded as belonging to a transition period in between the first and the second, not only chronologically but also from a conceptual point of view. It is already possible to feel a change of opinion about the foundations of mathematics. However the ideas are still not well shaped and one encounters almost incompatible remarks. The work that clearly marks that a change has happened is *Über das Unendliche* (1925).

Consider the following definition of axiom that reflects the effect of this shift and shows how the transition between the first and the second period is continuous.

In order to investigate a subfield of a science, one bases it on the smallest possible number of principles, which are to be as simple, intuitive, and comprehensible as possible, and which one collects together and sets up as axioms. Nothing prevents us from taking as axioms propositions which are provable, or which we believe are provable²².

In the above quotation the axioms, although are still basic principles, can just be provable propositions, or maybe-provable propositions.

In the same paper, later on, the concept of axiom is defined in the following ways:

The continuum of real numbers is a system of things which are linked to one another by determinate relations, the so-called axioms²³.

The following definition is also present in *Die Grundlagen der Mathematik* (1928) and in *Über das Unendliche*,

Certain formulas which serve as building blocks for the formal structure of mathematics are called *axioms*²⁴.

First of all we need to notice that the axioms do not define any kind of mathematical objects, but just their relations²⁵. On the other hand the difference between axioms and other formulas begins to be less marked. Hilbert says:

²²[Hilbert 1922].

²³[Hilbert 1922].

²⁴[Hilbert 1922], [Hilbert 1925].

²⁵As we noted in the previous paragraph this change in Hilbert's conception of axioms may be traced back to its correspondence with Frege.

The axioms and provable theorems [...] are the images of the thoughts that make up the usual procedure of traditional mathematics; but they are not themselves the truth in any absolute sense. Rather, the absolute truths are the insights that my proof theory furnishes into the provability and the consistency of these formal systems²⁶.

In the above quotation one can see not only that axioms and provable propositions have the same importance, as far as they are “images of the thoughts”, but also that axioms are deprived of their absolute truth²⁷ and keep just an operational character for the “usual procedure of traditional mathematics”. It is also possible to read the beginning of a separation between the concepts of consistency and truth, as we will explain later considering the further development in the thirties.

During the Twenties Hilbert’s proof theory was born. It was based on the the following ideas:

1. The whole of mathematics is formalizable, so that it becomes a repository of formulas.
2. There exists a *metamathematics* capable of handling in a meaningful way the meaningless formulae of formalized mathematics.

Consequently the axiomatic method becomes a tool to discover the principles of the whole mathematics in its formalized representation and to organize this representation. Recall that, in the earlier period, the axiomatic method consisted in the analysis of a field of mathematical knowledge, in order to isolate the principles and to make such a field a formal theory.

In this new perspective Hilbert defines a new kind of axiom.

This program already affects the choice of axioms for our proof theory²⁸.

These new axioms are not of the same nature as the ones mentioned in the previous quotations. They are the axioms on which the mathematical building rests. These axioms are logical and arithmetical in characters and are true axioms, in an absolute sense, since they draw their certitude and evidence from how Hilbert is now setting the problem of the foundation of mathematics: a proof theory that tries to justify the ideal elements with finitary tools.

²⁶[Hilbert 1923].

²⁷Not that Hilbert considered axioms as absolutely true but previously he assigned them the function of defining concepts.

²⁸[Hilbert 1923].

This circumstance corresponds to a conviction I have long maintained, namely, that a simultaneous construction of arithmetic and formal logic is necessary because of the close connection and inseparability of arithmetical and logical truth²⁹.

Since metamathematical statements have a content (*Inhalt*), intuition cannot be ignored in the foundation of mathematics. Since we want to postpone to the next section the analysis of the concept of intuition, now it is enough to say that intuition is the source of certainty and evidence for mathematics and it is capable of making mathematical truths absolute. Intuition is the origin of certainty in the finitary setting. So Hilbert, considering the whole mathematics as a complex of formal propositions, founds the certainty of mathematics in the intuitive relationship between the thinking subject and the symbols, “immediately clear and understandable”.

Following the terminology of Feferman³⁰, we could call Hilbert’s logical-arithmetical axioms *foundational* (while the axioms of a non foundational theory can be called *structural*).

Finally in 1925,

Certain of the formulas correspond to mathematical axioms. The rules whereby the formula are derived from one another correspond to material deduction. Material deduction is thus replaced by a formal procedure governed by rules. The rigorous transition from a naïve to a formal treatment is effected, therefore, both for the axioms (which, though originally viewed naïvely as basic truth, have been long treated in modern axiomatics as mere relations between concepts) and for the logical calculus (which originally was supposed to be merely a different language)³¹.

Here we see Hilbert’s full awareness of the changed concept of structural axioms. Indeed for Hilbert axioms were “originally viewed naïvely as basic truth”, as Euclid did, then at the beginning of this new axiomatic era as “mere relations between concepts”, like he himself did in the *Grundlagen der Geometrie*. Finally Hilbert thinks that his proof theory brought to the end this process of formalization of mathematics, so that also the structural axioms need to be viewed as meaningless formulas and they do not have more meaning than other mathematical propositions.

It is important to notice that in this second period the axiomatic method is considered as a logical tool, subordinate to a safe foundation of mathematics to be pursued by means of proof theory. The axiomatic method is no more than a tool for the analysis of a scientific theory also outside mathematics.

²⁹[Hilbert 1922].

³⁰[Feferman 1999]

³¹[Hilbert 1925].

Today there is a general method for the theoretical treatment of the questions in the natural sciences, which in every case facilitates the precise formulation of the problem and helps prepare its solution – namely the axiomatic method³².

The working of the method has to be logical. Hilbert says: “The axiomatic method pertains to logic³³”. We could read here a sort of logicism in Hilbert’s ideas, but this would be a mistake. Indeed the axiomatic method is not the tool capable of giving foundation to mathematics, but it is only used to formalize mathematics, like a preparatory study. This preparatory work puts the mathematician in the position of applying proof theory. We thus have an instrumental conception of logic, not a foundational one.

Let us now analyze the concept of intuition. We anticipate that we will find two different concepts of intuition and that this difference is responsible for two different concepts of axiom.

3 Hilbert and intuition

The intuitive character of the axioms is what makes the difference between them and other true propositions.

First of all we need to stress the difference between “intuitive” and “evident”, since the confusion between these two concepts has always been source of ambiguity. By “evident sentence” we mean a sentence that does not need to be analysed to exhibit its truth. By “intuitive sentence” we mean a sentence whose truth, in a given context, is immediately perceived, so that it is possible to skip some step of reasoning that, in other cases, would be necessary. Unlike evidence, which is innate within our mind, intuition can be educated thanks to use and mathematical practice. There is an important *caveat* though. The intuition we are talking about, that we could call a contextual intuition, is not an intuition that depends on a specific faculty of the mind, different from reason. In other words it is not a Kantian-style intuition, i.e. a faculty linked to the human perceptive structure that allows to perceive the aprioristic aspects of sensible knowledge (as the pure intuition of space and time is). This latter faculty is not intellectual, since it acts in the act of perception and makes perception possible.

Moreover we should also distinguish two modes of intuition; following [Parsons 1995] we call them intuition *of* and intuition *that*, to stress the difference between the conception of intuition as a kind of perception (*à la Gödel*³⁴) and the idea that intuition can be a propositional attitude. In

³²[Hilbert 1930].

³³[Hilbert 1930].

³⁴In [Parsons 1995] Parsons shows that this kind of intuition, although is explicitly defended by Gödel in [Gödel 1964], is not the only one that can be found in Gödel’s works. Starting from this right remark, it would be interesting to analyze the analogies

neither cases intuition is a form of knowledge. What any kind of intuition lacks to become knowledge is the evident characters that make existent (in same sense) the objects of intuition and true the propositions intuited. This kind of knowledge is direct and not mediated by a rational process. In the context of mathematics, this means that we can have knowledge of the existence of mathematical object or knowledge of the truth of mathematical propositions without a proof.

We will see how these different concepts can determine the nature of axioms.

3.1 First period

In the earlier Hilbert's foundational works the context of a mathematical theory plays a fundamental role in the choice of the axioms. For this reason the "axiomatic investigation of their [i.e. of the signs] conceptual content" is relative to a given theory and allows the "use of geometrical signs as a means of strict proof". Indeed Hilbert thinks that "the use of geometrical signs is determined by the axioms".

A precise account of mathematical signs is now outlined, together with a definition of axioms and their link to proofs and intuition.

Mathematical signs, including geometrical figures, can be used in a proof as far as their conceptual content is adequate to context, that is when signs formalize principles that are coherent with the basic concepts of the theory. Then they can be used as demonstrative tools, in the ways allowed by the axioms. The "conceptual content" is just the meaning of signs. This meaning depends on the axioms, since they define "certain [...] fundamental facts of our intuition", which are the base of an axiomatic system.

To new concepts correspond, necessarily, new signs. These we choose in such a way that they remind us of the phenomena which were the occasion for the formation of the new concepts. So the geometrical figures are signs or mnemonic symbols of space intuition and are used as such by all mathematicians³⁵.

As described by Hilbert, the process of axiomatization starts from an intuition concerning a domain of facts (*Tatsachen*), then, while formalizing it, it tries to clear the logical relationships within the concepts of the theory. The process, as Hilbert says, from the subject matter of a theory leads to a conceptual level³⁶.

and differences between the (at least) two different conception on intuition in Gödel's thought in comparison with Hilbert's.

³⁵[Hilbert 1900].

³⁶Recall that at the beginning of [Hilbert 1899] Hilbert quotes Kants' *Kritik der reinen Vernunft* and writes "All human knowledge begins with intuitions, thence passes to concepts and ends with ideas". This quotation, though not Kantian in spirit, explains how Hilbert wanted to use the axiomatic method in his researches.

The method of the axiomatic construction of a theory presents itself as the procedure of the mapping [*Abbildung*] of a domain of knowledge onto a framework of concepts, which is carried out in such a way that to the object of the domain of knowledge there now correspond the concepts, and to statements about the objects there correspond the logical relations between the concepts³⁷.

The signs then correspond to the images of the concepts one has in mind when working in the domain of knowledge. By deepening the foundations of a domain of knowledge one elucidates, at once, the logical structure of the theory and the intuitions about the subject matter of the theory.

Therefore axioms have a double role with respect to signs. On the one hand axioms, through the axiomatic enquiry, are used to give meaning to signs, on the other hand they grant the demonstrative power of signs, linking intuition to mathematical practice. Indeed intuition both precedes axiomatization and guides the work of mathematicians.

[O]ne should always be guided by intuition when laying things down axiomatically, and one always has intuition before oneself as a goal [*Zielpunkt*]. Therefore, it is no defect if the names always recall, and even make easier to recall, the content of the axioms, the more so as one can avoid very easily any involvement of intuition in the logical investigations, at least with some care and practice³⁸.

The kind of intuition that is in action with axioms and that allows to give meaning to mathematical propositions, is not evidence, but it is a contextual intuition that develops in parallel with the demonstrative techniques. It is the same intuition according to which mathematicians isolate and choose the axioms of a theory³⁹. It is the intuition that one develops when working within a theory. The axiomatic method then consists in formalizing, by means of signs (figures, symbols or diagrams), a *modus operandi* acquired by habit. Following the terminology fixed before it is an intuition *that*: a propositional attitude towards mathematics, that can be formalized and gain certainty, once a consistency proof is given for the formal system that

³⁷[Ewald and Sieg 2008].

³⁸[Hilbert *1905].

³⁹When working on the foundation of geometry Hilbert explains his aims in the following way: “we can outline our task as constituting a *logical analysis of our intuition [Anschauungsvermögens]*” and he explicitly talks about spatial intuition, but “the question of whether spatial intuition has an *a priori* or empirical character is not hereby elucidated”. Nevertheless the quotation above (from [Hilbert *1905]) shows that the intuition involved is not just a faculty of sensation. As a matter of fact, in these years, there is no philosophical analysis of the faculty of intuition.

embodies its syntactic counterpart: the signs. It is not an innate intuition, but it is sufficiently reliable to be used as an heuristic criterion and that can be formalized, once it is shown to be correct. Obviously this criterion is not always safe:

[...] we do not habitually follow the chain of reasoning back to the axioms in arithmetical, any more than in geometrical discussions. On the contrary we apply, especially in first attacking a problem, a rapid, unconscious, not absolutely sure combination, trusting to a certain arithmetical feeling for the behavior of the arithmetical symbols, which we could dispense with as little in arithmetic as with the geometrical imagination in geometry⁴⁰.

Even if it seems to be clear that for Hilbert the meaning of mathematical sentences depends on the intuitive content of the signs that compose them, the problem of the adequacy of the logical structure of a theory to the concepts it formalizes is left open⁴¹. The following quotation makes evident how problematic is for Hilbert the relationship between a formal system and the reality.

In physics and nature generally, and even in practical geometry, the axioms all hold only approximately (perhaps even the Archimedean Axiom). One must however take the axioms precisely, and then draw the precise logical consequences, because otherwise one would obtain absolutely no logical overview⁴².

In this is quotation the lack of an exact correspondence between nature and mathematics is indeed the reason why one needs to aim at the conceptual level⁴³. Contrary to what Galileo thought about it, there is no logical structure of the nature, but it is the task of science to succeed in imposing a logical form to the vague matter.

It seems that Hilbert does not give an answer to this problem in the early papers. Later he will try to solve it in the second period explicitly endorsing a Kantian-style notion of intuition.

⁴⁰[Hilbert 1900].

⁴¹Leaving aside the metaphysical problem of what is adequate to what, the simplest case of this question is the problem of the applicability of formal knowledge to nature. As a matter of fact, since meaning is contextual to formal systems, how are these able to formalize intuitions coming from the external realm of nature? The Kantian view which Hilbert will endorse during the second period tries to solve also this problem. In [Hilbert 1930], Hilbert will say: “*We can understand this agreement between nature and thought, between experience and theory, only if we take into account both the formal element and the mechanism that is connected with it.*”

⁴²Note on the front cover of Hilbert’s *Ausarbeitung* of the notes “Elemente der Euklidischen Geometrie”, in [Hallet and Majer 2004].

⁴³This remark is well explained in [Hallet 2008].

All these remarks show that at the beginning there is no coincidence between the notion of intuitive and the notion of evident. Indeed, Hilbert's explicit purpose, while writing the *Grundlagen der Geometrie*, was to give a safe basis to geometry different from intuition, unlike the euclidean axiomatic setting. Hilbert wanted to justify also non-euclidean geometries, so, after refusing evidence as a criterion for truth, he looked for a sufficiently general and comprehensive principle to give foundation to geometry, i.e. the axiomatic method⁴⁴. Nevertheless signs need meaning, in order to avoid a meaningless discourse. This is the “conceptual content” mentioned by Hilbert, where the intuition that gives meaning to signs is not the pure intuition of space (in case of geometry), but it is the intuition of the basic concepts of the theory that are formalized by means of axioms. This intuition is contextual to the formal system, it is the intuition that allows us to determine, thanks to the axioms, what are points, lines and space, as far as they are geometrical entities, i.e. part of a geometrical formal theory. In this first period a consistency proof carries also the burden of making possible mathematical knowledge, since it has to be knowledge of something true and existent.

3.2 The second period

In the Twenties, when engaged in the foundations of mathematica, Hilbert's new conception of axiom mirrors a deeper enquiry about the concept of intuition, in the direction of a Kantian-style notion. Thanks to that Hilbert thought to have solved the problem of a safe foundation for mathematics.

We start with two quotations which sound very Kantian.

Instead, as a precondition for the applications of logical inferences and for the activation of logical operations, something must already be given in representation⁴⁵: certain extra-logical discrete objects, which exist intuitively as immediate experience before thought. [...] Because I take this stand point, the objects of number theory are for me [...] the sign themselves, whose shape can be generally and certainly recognized by us — independently of space and time, of space and time, of the special conditions of the production of the sign, and of insignificant differences in the finished product⁴⁶.

Kant taught [...] that mathematics treats a subject matter which is given independently of logic. Mathematics therefore can never be grounded solely on logic. [...] As a further precondition

⁴⁴Even if, at the time, Hilbert lacked the logical tools.

⁴⁵In German *in der Vorstellung*.

⁴⁶[Hilbert 1922].

for using logical deduction and carrying out logical operations, something must be given in conception, viz. certain extralogical concrete objects which are intuited as directly experienced prior to all thinking⁴⁷.

In these quotations Hilbert's Kantism is clearly outlined. As Kant did before, Hilbert tries to give a foundation to the certainty of mathematical truths, not by means of logic, but reflecting on the very possibility of any mathematical knowledge. For Kant the a priori conditions of any knowledge were the pure intuitions of space and time. Hilbert, going further in the same direction, gives a foundation to certainty of mathematical truths by means of the sensible pure intuition of mathematical signs. Kant thought that spatial intuition was necessary for geometrical knowledge and temporal intuition was necessary for arithmetical knowledge. Hilbert thinks that the intuition of mathematical symbols (sensible intuition, since it is written on some physical support, and pure, since it does not depend on the shape the signs assume) is necessary to make knowledge within a formal framework possible. So, since every piece of mathematics is formalizable, symbols are pre-conditions of any form of mathematical knowledge.

Hilbert's purposes are clearly Kantian. It remains to be seen how much Hilbert's ideas towards the realization of those purposes are really Kantian. The affinity of the two thinkers looks in fact merely verbal and, maybe, for Hilbert functional to philosophers' approval. Indeed at that time the forms of neo-Kantism were quite spread and often quite far from Kant's original ideas.

The work *Naturerkennen und Logik* (1930) is the opportunity for a deep reflection on Hilbert's part on the philosophical meaning of this new conception. First of all Hilbert explicitly says that the older conception of the axiomatic method, offered at the beginning of the century, is not sufficient:

How do matters stand with this axiomatics, which is today on everybody's lips? Now, the basic idea rests on the fact that generally even in comprehensive fields of knowledge a few propositions – called axioms – suffice for the purely logical construction of the entire edifice of the theory. But their significance is not fully explained by this remark⁴⁸.

It follows an analysis of the sources of human knowledge. Hilbert claims that these are not just the intellect and the experience, but there is a third way: the *a priori* knowledge.

Whoever wishes nevertheless to deny that the laws of the world come from experience must maintain that besides deduction and

⁴⁷[Hilbert 1925].

⁴⁸[Hilbert 1930].

experience there is a third source of knowledge. Philosophers have in fact maintained – and Kant is the classical representative of this standpoint – that besides logic and experience we have a certain *a priori* knowledge of reality⁴⁹.

Now Hilbert argues in favour of an *a priori* ground for mathematical knowledge and he shows how his enquiry is coherent with this idea. In full accordance with “the most general and fundamental idea of the Kantian epistemology [...]: namely the philosophical problem of determining the intuitive, *a priori* outlook and thereby of investigating the condition of possibility of all conceptual knowledge and of every experience⁵⁰”.

Hilbert then shows how the distinction between *a priori* and *a posteriori* needs not be drawn at the level of the concepts of space and time. Indeed the works of Riemann, Helmholtz and Gauss, in the field of geometry, and the work of Einstein, in physics, have shown that space and time are not absolute concepts. So Hilbert, identifying *a priori* and absolute, draws the conclusion that space and time are not *a priori* concepts. Then

We therefore see: the Kantian theory of the *a priori* still contains anthropological dross from which it must be liberated; afterwards only the *a priori* attitude is left over which I have characterize in several works⁵¹.

If Hilbert had said that mathematicians have intuition (*à la Kant*) of the logical structure of formal systems, then there would have been a substantial coincidence between the first and the second period⁵². However Hilbert saw the end of Frege’s logicist program and was disappointed by the work of Russell e Whitehead, and for these reasons he thought that mathematics could not find a safe foundation on the ground of logic only. Moreover he wanted to argue against intuitionism on its own ground.

The intuition that allows to perceive the *a priori* character of the axiomatic setting is, according to Hilbert, the pure and sensible intuition of mathematical symbols that is deeply linked to the finitary method.

Hilbert’s aim is to give a foundation to mathematics on the clear perceptions of symbols. The fact that it is possible to perceive them, together with “their properties, differences, sequences and contiguities⁵³” makes elementary arithmetic possible. Then arithmetic, together with logic, gives a foundation to mathematics.

William Tait, in [Tait 2010], showed with clear arguments that Kant’s intuition is intuition *of*, since it is active in the process of perception. As far

⁴⁹[Hilbert 1930].

⁵⁰[Hilbert 1930].

⁵¹[Hilbert 1930].

⁵²See the thesis of Laserna in [Laserna 2000].

⁵³[Hilbert 1925].

as this kind of intuition is concerned, also Hilbert’s conception of intuition, in the second period, is of the same kind, since it is sensible and pure. Nevertheless there is an important difference here between the two thinkers in what concerns the aspects of evidence linked to this mode of intuition. Indeed Hilbert was thinking about the intuitive knowledge that we can find in the mathematical reasoning: “also [...]mathematical knowledge in the end rests on a kind of intuitive insight [*anschaulicher Einsicht*]”⁵⁴. On the contrary intuition, for Kant, is not a kind of knowledge, since in the intuitive process lacks the concepts under which the objects, given in the intuition, fall.

This idea of intuition and of our handling of mathematical symbols determines the foundational axioms to be assumed for proof theory. As a matter of fact they formalize the “fundamental elements of mathematical discourse⁵⁵”, that are, for Hilbert, pre-conditions of any knowledge of a formalized discourse.

The fundamental idea of my proof theory is none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds⁵⁶.

Hilbert then goes a step further: he claims that our intuitions of symbols have not only an *a priori* character, but they also manifest typical features of evidence.

The subject matter of mathematics is, in accordance with this theory [i.e. the accordance between symbols and our perceptive structure], the concrete symbols themselves whose structure is immediately clear recognizable⁵⁷.

This is the main difference between the first and the second period, and also the main difference between Kant’s and Hilbert’s intuition. In the first period intuition and evidence are kept apart, in the second one they coincide, thanks to a Kantian-style (for Hilbert) notion of intuition, that is like Leibniz’s intuitive knowledge in [Leibniz 1684], that, in [Parsons 1995], is described in the following way:

Knowledge is intuitive if it is *clear*, i.e. it gives the means for recognizing the object it concern, *distinct*, i.e. one is in a position to enumerate the marks or features that distinguish of one’s concept, *adequate*, i.e. one’s concept is completely analyzed down

⁵⁴[Hilbert 1930].

⁵⁵[Hilbert 1923].

⁵⁶[Hilbert 1928].

⁵⁷[Hilbert 1925].

to primitives, and finally one has an immediate grasp of all these elements⁵⁸.

It is easy to see that Hilbert thought that, thanks to his proof theory and the individuation of the *a priori* character of the intuition of mathematical signs, mathematical knowledge meets all these requirements. Moreover Hilbert thinks that this kind of knowledge makes basic arithmetic safe and that it is possible to extend it to the transfinite domain, assuming consistency of the formal system that incorporate mathematics. Logical tools are just, as Kant said, harness of the reason. Hence, in this second period symbolic logic is a tool used for a complete deployment and correct use of intuition. Moreover intuition, since it coincides with evidence, is able to give knowledge and certainty to formalized propositions.

It is important to stress that the intuition underlying Hilbert's foundational studies, at the time of the discovery of the proof theory, even if it is an intuition *of*, does not witness an evolution towards a stronger realism in Hilbert's thought, nor a shift towards a greater role of the philosophical roots of his investigations. Curtis Franks⁵⁹ has recently pointed out⁶⁰ the autonomy of mathematics in Hilbert's program. The point we would like here to make is that the intuition *of* in this later period is not a philosophical shelter from mathematical problems. It is not intuitions of the numbers, whatever they are. It is intuition of the accordance between the formalization of arithmetical and logical concepts, by means of the signs, and the concepts themselves. This accordance is given, on one hand by the perception of the signs and on the other hand by the awareness of the fact that the way we use signs mirrors how we are used to reason about arithmetical and logical concepts. In other words Hilbert's foundational effort is not ontological, but epistemological in character.

To sum up, at the base of two different concepts of axiom there are two different conceptions of our intuitive relation with symbols. Initially axioms define basic concepts of theories. So the content of symbols depends on the axioms not only since they define, or at least determine, concepts formalized by symbols, but also because axioms determine the use of symbols for mathematical practice. Then axioms have both a definitory and an operational function and their choice depends on a contextual intuition that is used to isolate the basic principles of a theory. It is intuition *that* and, thanks to a consistency proof, is the content of mathematical knowledge.

Later, at the time of Hilbert's program, evidence and intuition are identified and this coincidence is made apparent in the perception of mathematical symbols. The finitary point of view, together with Hilbert's proof theory,

⁵⁸[Parsons 1995].

⁵⁹[Franks 2009].

⁶⁰We do not agree with Franks that this is the only aspect of Hilbert's foundational work and we hope that what follows will show the reasons of our disagreement.

is based on this intuition. Intuition and evidence of mathematical symbols can determine the a priori principles of mathematical reasoning, in its formalized framework, and hence the choice of the logical-arithmetical axioms. The intuition described in this period is then intuition *of* and thanks to a consistency proof mathematical knowledge can be extended from the finitary to the transfinite domain.

4 Hilbert's program, Gödel's theorems and set theory

We have now to consider the impact of Gödel's theorems, in the light of the distinction between the first and the second period of Hilbert's enquiry. As we tried to show, what was at stake was non only the conception of axioms, but also the fiddly muddle of relations among axioms, theorem and proofs, that, according to Hilbert, have to be defined within a formal system.

If by "Hilbert's program" we refer to the attempt to give a consistency proof, by means of finitistic methods, of the formal system of infinitary mathematics, then we can say that substantially, accepting Church-Turing's thesis, Gödel's second incompleteness theorem marks the end of this program.

This wasn't Gödel's point of view in 1931, at the time of the discovering of the incompleteness phenomenon. Even if Gödel, in [Gödel *1933o] and in [Gödel *1938a], holds a substantial identification between *PRA* and Hilbert's finitism, in [Gödel 1972], he says that "Due to the lack of a precise definition of either concrete or abstract evidence there exists today, no rigorous proof for the insufficiency (even for the consistency proof of number theory) of finitary mathematics". Indeed Hilbert said what the finitary point of view was, but he didn't explain how to use it in proving the consistency of number theory. Nevertheless at the level of the basic ideas of Hilbert's proof theory, Gödel himself is aware of the fact that the kind of intuition that Hilbert used to found mathematical knowledge was not sufficient:

Since finitary mathematics is defined as the mathematics of *concrete intuition*, this seems to imply that *abstract concepts* are needed for the proof of consistency of number theory ... By abstract concepts, in this context, are meant concepts which do not have as their content properties or relations of *concrete objects* (such as combinations of symbols), but rather of *thought structures* or *thought contents* (e.g. proofs, meaningful propositions, and so on), where in the proofs of propositions about these mental objects insights are needed which are not derived from a reflection upon the combinatorial (space-time) properties

of symbols⁶¹.

In this quote we can see that, even if Gödel tried to clear the concept of finitism in the sense of what it should be, he thought that the finitary point of view given by Hilbert, by means of the intuitive relationship between the subject and mathematical symbols, was not adequate to give a concrete definition of finitism. In other words even if Gödel did not abandon the idea that it could exist a meaningful and useful sense that could be given to the expression “finitary methods”, he thought that Hilbert’s philosophical explanations were not acceptable.

However, in Hilbert’s earlier conception we find a notion of formal system sufficiently dynamic that we can claim that Hilbert’s foundational program could partially survive.

In 1900 Hilbert was explicit in considering axiomatic systems as open systems.

If geometry is to serve as a model for the treatment of physical axioms, we shall try first by a small number of axioms to include as large a class as possible of physical phenomena, and then by adjoining new axioms to arrive gradually at the more special theories⁶².

The process of enquiry of the foundations of a theory does not end with the choice of a given system of axioms. On the contrary the process is always evolving in the direction of a better understanding of the basic concepts of a theory. These concepts are defined by the axioms, but it is possible that our experience⁶³ can lead us to widen the definitions and consequently the axiomatic system. In the same way Newtonian mechanics is a particular case of the more general relativity theory.

As it was hinted before, it is not easy to see how this idea of open-ended formal system can agree with axioms as implicit definitions; nevertheless it is a fact that this idea survives also at the beginning of the second period. In the works in between *Neubergründung der Mathematik. Erste Mitteilung* and *Die logischen Grundlagen der Mathematik* formal systems are still considered as open system, but axioms have already lost their fundamental conceptual role among mathematical propositions. This would be in favor of the persistence of an interpretation of the concept of axiom with fenomenological flavor, in Hilbert’s thought.

⁶¹[Gödel 1972]. We don’t try here to clear what material intuition is for Gödel, neither what finitism should be for him. What we are interested in is what Gödel thought about Hilbert’s philosophical ideas behind the finitary proposal.

⁶²[Hilbert 1900].

⁶³Here experience is to be intended in the widest sense possible. It is not a sensorial experience (akin to the platonic one), but it rather is the progress of a theory, that by adding new theorems deepens into the understanding of the basic concepts of a theory.

Thus the concept “provable” is to be understood relative to the underlying axioms-system. This relativism is natural and necessary; it causes no harm, since the axiom system is constantly being extended, and the formal structure, in keeping with our constructive tendency, is always becoming more complete⁶⁴.

The process of extending a formal system, made necessary by Gödel’s theorem, never stops. As a matter of fact there will never exist a formal system sufficiently powerful to be able to develop arithmetic and at the same time to be deductively complete.

Still the conception of the axiomatic method as a tool to deepen the understanding of the basic concepts of a theory, contrary to Hilbert’s program as outlined above, survive after Gödel’s theorems. Indeed the development of always stronger⁶⁵ formal systems, capable of showing the consistency of the weaker ones, is what gave rise to so called *Gödel’s program*⁶⁶ in set theory.

This program aims to extend *ZFC*, the first order formalization of set theory, with new axioms, in order to find a sort of limit completeness that can never be reached. The goal of this program is in a sense the same as Hilbert’s program, i.e. to remove any *ignorabimus* from mathematics. Since we cannot have completeness by means of a single formal system Gödel’s program aims to find the right formal context where a solution of a given problem can be found. Gödel was hoping that the independence phenomenon was just a problem of our deductive tools, not a property of a specific class of problem.

In Gödel’s words:

It is well known that in whichever way you make [the concept of demonstrability] precise by means of a formalism, the contemplation of this very formalism gives rise to new axioms which are exactly as evident as those with which you started, and that this process can be iterated into the transfinite. So there cannot exist any formalism which would embrace all these steps; but this

⁶⁴[Hilbert 1922]. The shift between the two periods is here to be seen in what it is that is “becoming more complete”; before were the concepts formalized within the formal system, but later is just the formal structure of the system. However in the second period what is formalized is mathematical knowledge and this is what becomes more and more complete in extending the axioms.

⁶⁵Here we are talking about consistency strength, i.e. about large cardinals. These cardinals form a linear order thanks to which we can compare the strength of different theories, as far as the theories are equiconsistent with some large cardinal hypotheses. For a detailed presentation of the subject see [Kanamori 1994].

⁶⁶As far as I know this terminology dates back to Feferman who explicitly call it in this way in [Feferman 1996], but it is now a standard terminology, not also among philosophers of mathematics (for example [Hauser 2006]), but also among working set theorists (see [Woodin 2001]).

does not exclude that all these steps [...] could be described and collected together in some non constructive way⁶⁷.

Similarly to Hilbert, Gödel in 1964 thought that

a complete solution of these problems [e.g. the continuum hypothesis] can be obtained only by a more profound analysis (than mathematics is accustomed to give) of the meaning of the terms occurring in them (such as “set”, “one-to-one correspondence”, etc.) and of the axioms underlying their use⁶⁸.

The analogy with set theory we propose is not only a metaphor, but it also has methodological aspects. In the next quotation, taken from *Die logischen Grundlagen der Mathematik*, Hilbert’s last work from the transition period, a methodology is outlined which can be extended to modern set theory. This method consists of a constant dialectics between the demonstrative moment and meta-theoretical analyses that lead to the adoption of new axioms. Indeed, leaving aside the purely formal aspects that characterize proofs at the metamathematical level, we can find in Hilbert’s conception of the axiomatic method the roots of the useful relationship, for the mathematical discourse, between theory and practice.

Thus the development of mathematical science as a whole takes place in two ways that constantly alternate: on the one hand we drive new provable formulae from the axioms by formal inferences; on the other, we adjoin new axioms and prove their consistency by contentual inference⁶⁹.

Finally the similarity between Hilbert and Gödel can be also seen in the criteria to be adopted in the choice of new axioms. First I quote Hilbert:

If, apart from proving consistency, the question of the justification of a measure is to have any meaning, it can consist only in ascertaining whether the measure is accompanied by commensurate success⁷⁰.

And then Gödel:

There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems (and even solving

⁶⁷[Gödel 1948]

⁶⁸[Gödel 1964]

⁶⁹[Hilbert 1923].

⁷⁰[Hilbert 1925].

them constructively, as far as that is possible) that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory⁷¹.

In conclusion we can say that Hilbert's foundational efforts, although they are motivated by mathematical problems such as the antinomies and they try to give solutions essentially mathematical in character to those problems, like the proof theory, they nevertheless face and bring truly philosophical questions. As a matter of fact, in Hilbert's works we can find ideas of how to solve the problem of objectivity of mathematics and explanations of how is possible mathematical knowledge. It is sufficient to read Hilbert's foundational writings to go far beyond the formalist label given to his conception of mathematics. It is then possible to find not only a deep and complex reflection centered on the concept of mathematical sign, but also a progress in the direction of a clarification of mathematical and philosophical problems. One lesson that we can learn from Hilbert experience is that the mathematical tools we have nowadays are not sufficient to answer some philosophical question. For this reason if we get to the philosophical roots that inspired Hilbert's foundational efforts we can find that, for what concerns the second period, the developments of logic in the 20th century refute Hilbert's hopes. Nevertheless the ideas and methods, that Hilbert perfected in the first period and that made possible a foundation for geometry and the theory of real numbers, are still actual and they shape, at least in the case of set theory, contemporary mathematical practice.

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