CHAPTER

Advancing Algebra

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> Since *Algebra surpasses all human subtlety and the clarity of every mortal mind, it must be accounted a truly celestial gift, which gives such an illuminating experience of the true power of the intellect that whoever attains to it will believe there is nothing he cannot understand.*
>
> G. *Cardano*

What Is Algebra? Most people, when they think of "algebra," think of solving equations, factoring polynomials, graphing functions, and other things they did with x and *y* in their high school algebra classes. Too many people, including some who got good grades in mathematics, may recall that algebra was the point at which mathematics stopped having much connection with the real world.

In the past, many students have gone through algebra memorizing "rituals" (12), those formal procedures for manipulating symbols that have lost their connection to the numbers they represent. Now, hand-held computers can relieve us of the graph-plotting, symbol-manipulating drudgery of doing algebra. We are free to spend more time thinking algebraically. As teachers, we can begin to illuminate the structure of the algebraic forest without asking students to climb every tree.

> Rituals often lose something in the translation:
> "When you multiply, you add exponents; when you divide, you subtract."
> Students who do not understand the meaning behind this ritual may write:
>
> $$3^2 - 3^3 = 9^s.$$

With this exciting opportunity comes a daunting challenge: We need to reconstitute the algebra curriculum, to shift the emphasis from doing to thinking. Consequently, we have to decide what it means to "think algebraically." We must identify

essential concepts and skills and determine, through research, how much conceptual development and skill practice students need to be successful at solving algebraic problems. The NCTM *Curriculum and Evaluation Standards* (32) point some general directions to be followed, but it will be the continuing task of textware authors and classroom teachers to map details of the journey from the algorithmic algebra of yesterday to the problem-solving algebra of tomorrow.

> In algebra we need to increase our emphasis on:
>
> - Real-world problems
> - Conceptual understanding
> - Computer-based methods
> - Structure of number systems
> - Matrices and their applications
>
> NCTM *Curriculum and Evaluation Standards* (32, p. 126)

Research can help us understand how students construct mature concepts and learn complex procedures. Research can suggest activities for the classroom that foster the connections that build understanding. Research can suggest ways of making algebraic ideas accessible to all students (15), so that algebra becomes part of the mathematical pump that propels students on to higher achievement instead of the filter that holds them back (34). Translating theoretical ideas from research into practical ideas for the classroom can help teachers teach more effectively and students learn more efficiently.

Historical Sketch. Formal research in the learning of algebra began early in the twentieth century, about the time that psychologists were developing objective methods for measuring intelligence, aptitude, and achievement. Mathematics was a popular vehicle for studying constructs like learning and memory because of the ease of scoring answers. Algebra was often used for studying advanced learning because so few people had yet to understand the subject!

From 1900 to 1930, research on the learning of algebra dealt primarily with the relative difficulty of solving various types of linear equations. In 1923, Thorndike (47) published *The Psychology of Algebra,* in which he applied his "bond" theory to the learning of algebra. Though most of his recommendations focus on such things as the amount of practice students need to acquire certain skills and how the practice should be distributed, Thorndike is credited with bringing a systematic approach to research in the learning of algebra, including a careful analysis of the nature of algebraic tasks.

~From 1930 to 1945, research in education declined, as the nation focused on issues of survival surrounding the Great Depression and World War II. After the war, research in algebra rebounded, as a new wave of behavioral psychologists refined some of the earlier methods of investigating skill acquisition. It was in the 1960s that mathematics educators, with academic backgrounds in higher mathematics and teaching experience in secondary schools, began to shift the focus of research toward conceptual understanding. Their work has been complemented by the efforts of cognitive psychologists to analyze the thinking processes involved in problem solving and the acquisition of complex algebraic skills.

> "Algebra to most learners ... is in large measure forming more or less particular bonds or connections, such as $a \times ab = a^2b$, $a(a + b) = a^2 + ab$, a means $1a$, $-a \times -b = +ab$, learning to operate several of these together as needed, organizing them further into more inclusive habits and insights, summing up what one has learned to do in rules, and thus gradually attaining a sense of what it is right to do with literal numbers and why." Thorndike et al. (47, p. 246)

In this chapter we will discuss the recent research in algebra from three perspectives—that of the learner, of the content, and of the teacher. As we identify major findings, we will propose ideas for teaching certain topics. We will conclude with some general implications for the algebra classroom.

Learning of Algebra

Two theoretical perspectives underlie most of the research on the learning of algebra of the past twenty-five years. Piagetian-style theories of *cognitive development* (see Chapter 1) have provided a theoretical framework for much of the research on conceptual understanding. Written instruments and semi-structured interviews, using nonroutine tasks—that is, tasks that are different from typical textbook exercises—are used to analyze students' concepts of function, equation, and variable. Sometimes tasks are administered using techniques inspired in part by Soviet research methodology, such as asking individual students to think aloud while working on a problem, having pairs of students work together on a problem, or conducting small-group teaching experiments in which tasks are devised to capitalize on responses to previous tasks. Researchers then analyze tapes and transcripts of working sessions to formulate conjectures about students' thinking.

Information processing provides a theoretical framework for much of the current research on skill learning in algebra. Here, researchers study data on such factors as response time or common errors in an effort to identify patterns and infer mental processes students use in carrying out certain algebraic procedures. Sometimes computer models of students' thinking processes are formulated, or computer programs called "intelligent tutors" are developed to remediate student errors (see Chapter 11).

Both cognitive-development and information-processing research can help us understand how students learn. Both kinds of research can alert us to obstacles that arise along the way. Before we consider the mental constructs students need to develop in algebra, let us consider some sources of difficulty they typically encounter.

Research suggests that impediments to learning tend to be of three types: (a) Some are inherent to the subject itself, (b) some are intrinsic to the learner, and (c) some are the unintended consequence of generally good teaching techniques. Most obstacles inherent to algebra stem from notational conventions or the complexity of concepts that arise with the use of letters as variables. Obstacles intrinsic to the learner

include such human foibles as the tendency to overgeneralize or to judge on the basis of superficial characteristics. The most common obstacle attributable to teaching is the incomplete mental construct that may result from considering a too-narrow range of simple, special cases of a given concept (46).

Many obstacles to the learning of algebra are inherent to the subject itself:

$$|x| = x, \text{ if } x \geq 0$$
$$|x| = -x, \text{ if } x < 0.$$

We will see examples throughout the next section of obstacles of various kinds. We should keep in mind that cognitive conflict is not necessarily bad for students; in fact, it is an important stimulus to learning. It is our responsibility as teachers to be aware of possible sources of conflict and alert students to differences, as well as similarities, among the various phenomena they study.

Content of Algebra

For convenience, we have divided the discussion of content research into the language, concepts, geometry, and rules of algebra. Generally, research studies reflect a range of ideas and do not fall neatly into a single category. Thus, we will analyze the research by areas of implications, rather than study by study.

Language of Algebra

Algebra is a language for describing actions on, and relationships among, quantities. As with any language, difficulties may arise from features of the language itself or in translating from one language to another. Within the language of algebra, most linguistic difficulties relate to variables and expressions; most translation difficulties arise in translating word problems into equations.

Variables. Research shows that students can work with variables without fully understanding the power and flexibility of literal symbols (51). Because variables operate much like the numbers of arithmetic, and because conceptually they resemble pronouns in ordinary language, most students can acquire some facility in routine algebraic manipulations. On the other hand, variables are different from numerals (e.g., variables can represent many numbers simultaneously, they have no place value, they can be selected arbitrarily), and they are different from words (e.g., variables can be defined in any way we wish and can be changed without affecting the values they represent).

> Changing x to y in the equation $2x + 3 = 15$ does not have the same effect as changing *he* to *she* in the sentence, *He was President of the United States.*

Variables are versatile, too. We use them as names for numbers or other objects, as discrete unknowns in equations, as continuous unknowns in inequalities, as indeterminates in polynomials, as generalized numbers in identities, as independent and dependent variables in functions, as parameters in formulas, and so on. Whether we look at variables from the viewpoint of the roles they play in algebra (48) or the ways that students operate with them (24), it is clear that the "concept of variable" is, in fact, *a* multifaceted idea.

Can you add to this list of ways we refer to literal symbols?

Unknown
Variable
Constant
Parameter
Generalized number
Name
Placeholder
Argument
Indeterminate

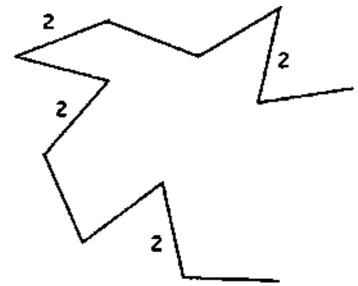
However, there is evidence that students' early impressions about variables may impede their construction of a sufficiently general concept. For example, students who first encounter variables as names (as in person A, person B) may assume that letters are like abbreviations. This assumption is reinforced when we use the mnemonic device of choosing as variables the first letter of the name of the objects we are talking about, as in using *a* to represent some number of apples. Interpreting variables as abbreviations may cause difficulty in translating word problems, as we will see shortly.

Students who first encounter variables as unknowns in equations may assume that letters represent a specific value. When confronted with a letter of indeterminate value, (see Fig. 7.1), they may respond by assigning a value to the variable and computing the answer (24).

A common naive conception about variables is that different letters must have different values. Even students who have been told and are quick to say that any letter can be used as an unknown may, nonetheless, believe that changing the unknown can change the solution to an equation (50). How many of us have unwittingly reinforced this misconception by always picking different values for *a, b,* and *c* to illustrate, say, the distributive law $a(b + c) = ab + act$ If we occasionally pick the same values for different variables, the reactions of our students could be instructive.

Expressions. An algebraic *expression* is a description of some operation involving variables, such as 3d, $x + 1$, or $x — y$. Because we use two distinct symbol systems (letters and numerals) together in algebra, and because these systems follow different rules, we gain some economy of notation, but at the expense of possible confusion. For instance, many students assume that $—x$ is a negative number (29), as though

FIGURE 7.1 N-gon diagram from CSMS



Letter used as a specific unknown.

Part of this figure is not drawn.

There are n sides altogether, each of length 2.

The perimeter p =____.

Adapted from (24, p. 23)

the variable were a numeral. In this particular case, we should alert students to the three uses of the "—" sign (the subtraction operation "minus," the integer sign "negative," the additive inverse "opposite of) and refer to "$—x$" as "the opposite of x" rather than "negative x."

The most common notational shortcut in algebra is the omission of the operation sign for multiplication. Because variables have no place value, we can denote multiplication by concatenating (linking) literal symbols with numerals, other literal symbols, or parentheses, as in 3d = 3 X *a* or 3(1/2) = 3 x 1/2. However, students are used to interpreting concatenation in arithmetic as implying addition (29), as in 32 = 30 + 2 or 3 1/2 = 3 + 1/2. It takes them awhile to internalize these conflicting conventions (10), and of course the fact that both arithmetic and algebraic conventions operate simultaneously in algebra does nothing to alleviate the confusion.

What do we get if we replace the a in 3a by 2 in arithmetic? What do we get in algebra? Adapted from (10), p. 41

One idea that should, but does not, transfer easily from arithmetic to algebra is the notion of inverse operations. In the elementary grades, a great deal of emphasis is placed on the inverse relation between addition and subtraction, multiplication and division. Yet, these inverse relationships tend to get lost in algebra. For example, most students can multiply $(3x + 2)(5x — 4)$ to obtain $15x^2 — 2x — 8$ readily enough, but if asked in the next breath to factor (the product) $15x^2 — 2x — 8$, surprisingly few students immediately recognize that the factors are the expressions they just multiplied (37). The complexity of algebraic operations can obscure important relationships, and often it is not enough just to remind students that multiplying and factoring are inverse operations—we need to illustrate the implications of a relationship with specific examples.

Just as young students who work with place-value ideas eventually begin to think of ten single units, bundled together, as a single unit of ten, so we expect algebra students who work with polynomial expressions to be able to "unitize" these expressions and treat them as single variables, as in factoring by grouping: $ax + bx + ay + by = (a + b)x + (a + b)y = (a + b)(x + y)$. The tendency of most students to think only of single letters as variables (2, 35, 52) may be partially the consequence of an algebra curriculum that uses only single letters as unknowns. Perhaps occasional use of unknowns like *In* or $x + 5$ might help students learn to unitize expressions.

Suppose $5(3z - 1) = 10$.

Then $\frac{3z - 1}{2} = ?$

How many of your students see the shortcut?

Adapted from (2)

Parentheses and other bracketing symbols should provide a perceptual aid for unitizing. However, there are three common behaviors that seem to neutralize the suggestive effect of parentheses. Some students apparently ignore or overlook bracketing symbols, as in $4(n + 5) = 4n + 5$ (6). Other students, perhaps in response to the order-of-operations exhortation to "do what's in parentheses first," focus on parentheses to the exclusion of the overall structure of the expression (39). And finally, the equation-solving advice to "clear the parentheses first" may prompt many students to overlook the variable unit in their haste to eliminate grouping symbols. As with any rules of thumb, we need to show students some situations in which the usual rules are not the most efficient way to proceed.

Word Problems. In their search for well-defined procedures to follow, students may transfer the "key word" approach to solving word problems in arithmetic to a "key context" approach in algebra. That is, they tend to categorize and remember problems according to superficial characteristics—as "distance" problems, "age" problems, or "mixture" problems—rather than according to underlying relations—equal quantities, two quantities added equal a third, and so on (20, 30). We may be able to help students focus more on underlying structures if we do as the *Standards* suggest and increase our use of real-world (not so easily categorized) problems and decrease the number of traditional coin-digit-work type problems (32, pp. 126-127).

> Word problems should be integrated throughout the chapter and not given as a separate assignment at the end. (28, p. 423)

Though looking for key words can be a useful problem-solving heuristic, it may encourage overreliance on a direct, rather than analytical, mode for translating word

problems into equations. The fact that so many textbook problems lend themselves to direct translation is seductive, and when faced with as simple a real-world situation as "There are six times as many students as professors," even college students translate the statement as $6S = P$ about half the time (40). Though direct translation is not always to blame, certainly the presence of the phrase "six times" and the interpretation of S as an abbreviation for *students,* rather than a representation for the *number* of students, impels many, including some who can draw an accurate diagram for the statement, to write an equation that resembles a literal translation but which is mathematically reversed. This reversal error seems highly resistant to remediation; however, the analysis required to write a computer program to provide the proper output has been moderately successful with some students (43).

A question that needs to be researched for word problems in general is whether using simple, non-mnemonic variables like x and y may wean students away from direct translation toward an analysis of numerical relationships. Moreover, having students write out in words what each variable represents ("*x* represents the number of students, *y* represents the number of professors") not only provides a visual reminder that the variables represent numbers, but is also very much in keeping with the increased emphasis on verbalization in the *Standards* (32, p. 140). The interested reader should see (9) for further review of factors related to word problems.

Concepts of Algebra

The two algebraic concepts that have been investigated the most are equations and functions. Conceptually, there is quite a jump from equations, in which a single variable typically represents one or two unknown numbers, to functions, in which two or more variables generally take on infinitely many values in relationship to each other. Inequalities are a conceptual intermediary between these two, in that a single variable represents a whole continuum of numbers, but relatively little research has focused on inequalities.

Equations. Students typically begin solving simple equations long before they enter a formal algebra course, but a clear vision of the structural differences between equations and expressions may be obstructed by their experience with the equal sign in arithmetic. Students do so much computing of answers in arithmetic, they may come to regard the equal sign as a kind of operation sign—a "write-the-answer" sign—rather than a statement of equivalence (4, 22).

In algebra, the equal sign may still signal writing an answer, as in simplifying an expression: $2x + 5 + 3x - 7 - 5x - 2$. But in solving equations, the equal sign is explicitly a relation sign, and students are asked to operate on the whole relation to find a sequence of equivalent relations. Lingering confusion between simplifying expressions and solving equations is betrayed when students refer to the (often numerical) right-hand side of an equation as "the answer" or when they simplify an expression, look at the "equation" they have thereby written, and begin solving it, only to wonder what happened to *x,* when all the *x* terms subtract out.

Have any of your students ever done this?

Simplify: $2x + 5 + 3x - 7$

Solution:
$$\begin{aligned} 2x + 5 + 3x - 7 &= 0 \\ 5x - 2 &= 0 \\ 5x &= 2 \\ x &= \frac{2}{5} \end{aligned}$$

Perhaps textbook authors should distinguish between the two uses of the equal sign in algebra by consistently using "≡" to denote identically equivalent expressions (axiomatic properties, simplified expressions, multiplication/factoring identities, equivalent equations, etc.) and " = " to denote the limited equality of an equation or function. Then, for example, students who are wont to solve equations by chaining successive, equivalent equations together with equal signs could quite properly write:

$$7x - 3 = 5x + 5 \equiv 2x - 3 = 5 \equiv 2x = 8 \equiv x = 4$$

Few students fully appreciate the fact that solving an equation is finding the value(s) of the variable for which the left- and right-hand sides are equal (52). Numerical approaches, such as using arithmetic identities to develop the concept of equation (19), or using calculator methods to find or approximate solutions (13), may help students focus more on the relational aspect of an equation and less on the algorithm for solution. Having students reason through a solution instead of always using inverse operations may also help (1; see Fig. 7.2).

Try this with your class:
Find an expression whose value is 17 when x = 3.
Possible response: 5x + 2.
Find another expression whose value is 17 when x = 3.
Possible response: 7x - 4.
Now, what is the solution of this equation:
5x + 2 = 7x - 4? How can you tell?
Adapted from (52)

FIGURE 7.2 Solving an equation by reasoning (1, p. 205)

Solve: $14 - \frac{15}{7 - x} = 9$

Reasoning: 14 minus what equals 9? (5)

15 divided by what equals 5? (3)

7 minus what equals 3? (4)

Solution: x = 4

Checking numerical solutions in *the original equation*—for word problems, in the problem statement—should be an integral part of the solution process for all types of equations (8), not just for rational or radical equations, where extraneous roots may be introduced. The significance of checking solutions to equations is not intuitively obvious to students (17). The purpose of checking is not just for accuracy, but also for ascertaining the reasonableness of an answer and reinforcing the connection between the original equation or problem situation and the final solution.

Functions. Some early research on concepts related to functions indicates that students construct the formal concept of function in stages, beginning with the notion of a function rule, and progressing through vocabulary and symbolism, graphical representation, operations on functions, and internal properties of specific functions (27). Much research has focused on intuitive ideas about functions and the transition from intuition to formal symbolism (e.g., 14, 36, 49). Research on graphing will be discussed in the next section.

Middle grades students can easily comprehend the basic idea of a function as "a rule of correspondence" either in concrete situations or in two-column tables of numbers. For simple functions, they can identify patterns, fill in missing domain/range elements, and verbally describe rules of association (7; see Fig. 7.3).

The formal $f(x)$ notation, on the other hand, condenses a great deal of information very efficiently but causes difficulty even for advanced students. In the Fourth Mathematics Assessment, for example, success rates of 11th graders evaluating $a + 7$ when $a = 5$ declined 20-40 percentage points (depending on algebra background) when the question was recast as evaluating $f(5)$ when $f(a) = a + 7$ (26, p. 62).

The set-theoretic definition of *function* that appears in many algebra textbooks does not convey the richness of the function concept in a very meaningful way. By and large, students' intuition about what constitutes a function corresponds more to the first functions they encounter than to the formal definition (49). That is, students generally believe that functions should be linear, or at the very least, continuous, smooth, and definable by a single formula. To help students construct a more com-

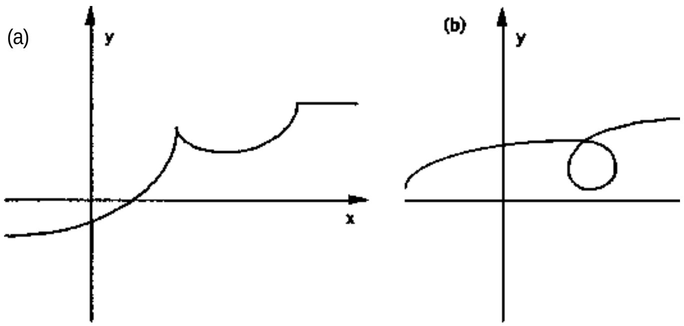
FIGURE 7.3 Item similar to one used in the Second Mathematics Assessment (7, p. 68)

x	y
1	6
3	
4	11
7	16
8	

What is y when x = 3? What is y when x = n?

FIGURE 7.4 Items similar to those used by [(49), p. 359]

Does there exist *a* graph whose function is:



plete conceptualization, we need to augment the introduction of simple examples with a variety of other kinds of functions, as well as nonexamples of functions (see Fig. 7.4). For more discussion of specific types of functions, see Chapter 9.

Geometry of Algebra

Traditionally, algebra and geometry have been regarded as separate subjects in the high school curriculum, with relatively few efforts to draw the connections between them that the *Standards* recommend (32, p. 146ff). Most of the geometry that has appeared in algebra classes has been as graphs of functions, and the treatment of this topic was limited by the practical difficulties of graphing by hand. Computers and graphing calculators enable us to take a much more visual approach to algebra.

Area Models. The Greeks developed algebraic ideas using geometry. Long before Viete systematized the symbolization of polynomials in the late sixteenth century, the Greeks proved polynomial identities using area models (16; see Fig. 7.5). General learning theory (see Chapter 1) suggests that pictorial representations, together with numerical concretizations,

$$29 = (20 + 5)^2 = 20^2 + 2(20 \times 5) + 5^2 = 400 + 200 + 25 = 625,$$

should help students apprehend the equivalence of algebraic identities in symbolic form. Research is needed to determine the most effective way of incorporating activities with area models into the algebra classroom.

Graphing. Prerequisites for graphing functions, such as notions of point and line, as well as plotting and naming points on coordinate axes, are standard fare in the elementary grades. Nevertheless, barely 50% of students who have studied algebra can do much more with coordinate systems than simply plot points (26, pp. 62-3). Even students who can graph a function like $x - 2y - 5$ may not be able to look

FIGURE 7.5 Geometric proof of the algebraic identity

$$(a + b)^2 = a^2 + 2ab + b^2$$

as found in Book II of Euclid's *Elements* (16)

at a given graph and identify a solution of the corresponding equation from the graph (52; see Fig. 7.6).

Some naive conceptions related to graphs seem to reflect naive conceptions in geometry, such as the common notion that the only points on the graph of a function are the points that were actually plotted (21). These kinds of errors can be at least partially attributed to the fact that most representations of geometric concepts have features contradictory to the concepts they represent. That is, representations of points all have size, whereas points themselves do not. When we plot points on a graph, they generally appear "bigger" than the points elsewhere along the line, and this perceptual miscue is misleading to many students. Calculator and computer graphing may alleviate some of these problems but may introduce new problems yet to be identified (46; see also Chapter 11).

FIGURE 7.6 Graph interpretation item from (52)

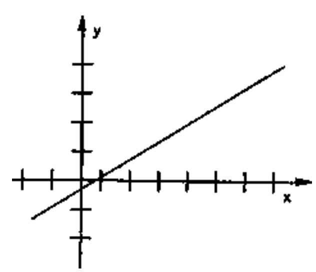


FIGURE 7.7 Graph of the speed of a vehicle with respect to time

Some misconceptions are not so much geometric in nature as they are idiosyncratic to graphing, as when students confuse the picture of the graph with the actual event (5). For example, it is difficult for students to imagine that a vehicle whose speed is graphed in Figure 7.7 could be going uphill when the graph of the speed is going down, and vice versa. A number of researchers (see [25] for a complete review) have asked students to construct stories whose actions could be depicted by a given graph (see Fig. 7.8). This kind of open-ended, multiple-answer problem solving, encouraged by the *Standards* (32, p. 137ff), is easily adaptable to the classroom and could enhance students' graph interpretation skills.

Rules of Algebra

Research on the rules of algebra focuses on describing errors and trying to explain thinking processes. The errors noted in the research literature are generally those familiar to every algebra teacher.

FIGURE 7.8 Graph with multiple interpretations. Students make up stories that could be illustrated by the graph.

Manipulating Expressions. Many errors related to expressions seem to involve an interaction between (a) overgeneralizing on the part of the student and (b) the highly abstract nature of the field properties, especially the associative or distributive law. For example, one of the most common errors is to simplify an expression like $4 + 3n$ to *In*. One explanation for this kind of error is that, in arithmetic, students learn to operate on numbers until they obtain a single number as the answer. Then, in algebra they may feel uncomfortable leaving an expression with a visible operation sign as the final answer, so they perform whatever operations they can on the available numbers to reduce the expression to a single term (11). Of course, in the case of $4 + 3n \rightarrow 7n$, if the first operation were multiplication instead of addition, combining the numbers would be correct, using the associative law:

$$4(3n) = 4\cdot(3\cdot n) = (4\cdot 3)\cdot n = 12\cdot n = 12n$$

Another common error is to simplify an expression like $\frac{x + 2}{2}$ to x by deleting the 2 in the numerator and denominator (8). Again, were the operation in the numerator multiplication instead of addition, dividing above and below by 2 would be correct: $\frac{x\cdot 2}{2} = x$. In the case of addition, the distributive law implies $\frac{x + 2}{2} = \frac{1}{2}\cdot(x + 2) = \frac{1}{2}\cdot x + \frac{1}{2}\cdot 2 = \frac{x}{2} + 1 \neq x$ (unless $x = 2$).

As a third example, the expression $(x + y)^2$ is often converted to $x^2 + y^2$, following the pattern of $(xy)^2 = x^2y^2$ (29). Of course, writing $(x + y)^2 = (x + y)(x + y)$ and using the distributive law (twice) helps clarify where the missing middle term $2xy$ comes from, in contrast to $(xy)^2$, which converts to x^2y^2 using only the commutative and associative laws.

The frequency and persistence of these kinds of errors clearly show that there is no easy answer to remediating them. Cautioning students to distinguish carefully between multiplication and addition (between factors and terms) undoubtedly helps but is not sufficient. We need to emphasize the importance of structural ideas in algebra by showing how the field properties apply, at the same time using the powerful technological tools at our disposal to clarify and reinforce the structural relationships with numerical examples and visual models.

> Standard 14: Mathematical Structure
> In grades 9-12, the mathematics curriculum should include the study of mathematical structure so that all students can … understand the logic of algebraic procedures. NCTM *Curriculum and Evaluation Standards* (32, p. 184)

Solving Equations. As with expressions, the errors in solving equations are familiar ones, perhaps the most common being variations of the sign error:

$$x + 37 = 150 \rightarrow x + 37 - 10 = 150 + 10$$

Interviews with students reveal that this error may not always be the result of carelessness or confusion with the transposition rule but may sometimes reflect a belief system that attributes validity to some operations that are not mathematically valid. In the above example, for instance, some students seem to believe in a fairness ("redistribution") principle: Whatever is taken away from one side of an equation should be added to the other side (23).

Just as some students seem more adept at constructing graphs than interpreting graphs already drawn, some students are more adept at solving equations than identifying given transformations that yield equivalent equations. In a recent study (44), about a third of the students who were asked to identify equivalent pairs of equations preferred to compute answers to determine the validity of transformations, even in simple cases like:

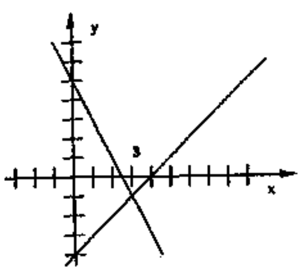
$$X + 2 = 5 \quad \text{and} \quad X + 2 - 2 = 5 - 2$$

Judging equivalence of equations was even more difficult when the transformation did not involve a number in the equation, that is, was not a usual step in a standard solution path:

$$X + 2 = 5 \quad \text{and} \quad + 2 - 99 = 5 - 99$$

Graphing calculators and recent software packages for algebra encourage students to solve equations by graphing the left-hand side and the right-hand side as functions and determining for what x-value the functions are equal (see Fig. 7.9). Students can then perform transformations on the equation and see whether the transformed functions still intersect at the same point or along the same vertical line (same x-value). Explorations of this sort are fun and instructive, and we may find that they help students better understand which transformations preserve the equivalence of equations and why.

FIGURE 7.9 Solving $x - 4 = -2x + 5$ by graphing each side of the equation as a function and finding the value of x for which the two expressions are equal



Teaching- of Algebra

As promised, we conclude with some general recommendations. Perhaps first and foremost, we need to incorporate more writing in the algebra classroom. The *Professional Teaching Standards* urge more writing throughout the mathematics curriculum (33, p. 52), but writing is especially important in a subject like algebra that is generally viewed as being highly symbolic, almost nonverbal. Journal writing at the beginning of class can provide a mental warm-up for students and improve their attitude toward mathematics (31). Writing during or at the end of class can identify questions, difficulties, or misunderstandings. As much as writing helps students, it can help the teacher even more by providing informal and highly individual feedback in a nonthreatening atmosphere. Students write the most in response to simple, but specific, prompts and particularly when asked to direct their comments to a parent, a friend, or a former teacher (31).

Writing need not be limited to journals, however. Essay test questions that ask students to explain a solution procedure to, say, a younger student not only enhance students' understanding through the process of articulating ideas, but provide the teacher with detailed information about students' conceptualizations. Although supplying students with worked examples of problems, from which to generalize a solution procedure, has met with only mixed success (38, 45), writing might be used in conjunction with incorrectly worked examples to provide students with a context for diagnosing errors and explaining correct procedures.

See if your students can find the error and explain the correct procedure:

Solve: $y^2 - 6y = -9$
Solution: $y(y - 6) = -9$
$y = -9$ and
$y - 6 = -9$ so $y = -3$

Research also shows the value of asking nonroutine questions to ascertain the depth of students' understanding. One technique that has often been used is simply to reverse the standard questions: Give students a graph and ask them to interpret it (5); give students an equation and ask them to make up a word problem for it (39); give students a solution and have them make up an equation or system of equations having that solution (52). Creative questions not only help the teacher/researcher by providing insight into students' thinking, but they also help students become less unidirectional in their thinking and to make those connections between questions and answers, concepts and processes that make learning more meaningful.

Special activities are good motivation for students. "Reversed" questions suggest a game of Jeopardy (3). "What's My Rule" helps clarify the idea of function (41). Bingo can be adapted for all sorts of algebraic activities.

Another theme that underlies much of the research is the importance of applications in algebra—from two perspectives. First, algebra gives us a way of modeling real-world phenomena and predicting outcomes through manipulation of abstract symbols. From this perspective, applied problems give meaning to abstract symbols. But from another perspective, the algebras of polynomials, rational expressions, and equations are themselves concrete examples of more abstract structures like rings, fields, and vector spaces. As such, the algebra of secondary school gives meaning to the abstract structures of higher mathematics. Stressing the importance of applied problem solving in algebra addresses only half of the applications equation. Emphasizing the structural aspects of algebra addresses the other half and balances the equation.

> "When using computing devices in the algebra classroom, choose software or tools that place mathematical decision making in the hands of the students … Tool software, such as function graphers, table generators, and symbolic manipulators, can put students in control of problem-solving decisions." (18, P- 197)

Of all the lessons we learn from research, perhaps the most important is that we must never take for granted that students truly understand a subject just because they are able to operate intelligently within it. We must be careful to explain the "obvious" and ask the questions that are not so obvious. Most of all, we need to listen to our students, to what they say and what they write. Learning, like teaching, is two-way communication. Students will learn more from us, the more we learn from them.

Looking Ahead . . .

Research in the learning and teaching of algebra is maturing to the point that it has important ideas to offer curriculum developers and teachers. New text materials, software programs, and standardized tests (42) are beginning to reflect recent ideas from research. Indeed, several of the writers of these new curriculum materials are researchers eager to share insights with students and teachers. As teachers become increasingly involved in research, communication is enhanced, and cooperation becomes collaboration. We will achieve full partnership when teachers and researchers are as equally involved in formulating the research questions as we are in searching for answers.

Sigrid Wagner

I believe that research can help teachers by providing suggestions for the classroom that help us teach so students will internalize ideas and avoid common misconceptions. Research will be of no use, however, if we do not try some of the ideas. We must heed the message of research into teacher effectiveness and adjust our planning, expectations, and behavior to create a classroom environment in which student input is at the center of every learning experience. I would suggest that every

teacher keep a journal that includes general feelings and interesting things that happen in the classroom. You may have questions that can only be answered in your own classroom. If we aren't always looking ahead to how we can become better teachers, then research will be of no value to us.

Sheila Parker

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